Some aspects of the spectral theory for $\mathrm{sl}\,(3,\mathbb{C})$ system with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ reduction of Mikhailov type with general position boundary condition

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Introduction. GMV and RGMV systems

In this work we shall consider the linear problem of the type

$$i\partial_x\psi + \begin{pmatrix} 0 & (\lambda-\lambda^{-1})u & (\lambda+\lambda^{-1})v \\ (\lambda-\lambda^{-1})u^* & 0 & 0 \\ (\lambda+\lambda^{-1})v^* & 0 & 0 \end{pmatrix}\psi = 0.$$

with boundary conditions

$$\lim_{x\to\pm\infty} u(x) = u_0, \quad \lim_{x\to\pm\infty} v(x) = v_0.$$

In the above u, v (the potentials) are smooth complex valued functions on x belonging to the real line and by * is denoted the complex conjugation. In addition, the functions u and v satisfy the normalizing relation $|u|^2 + |v|^2 = 1$. We call the above system the rational GMV (RGMV) system.

The problem of course has has its history. In series of papers authored by Gerdjikov, Grahovski, Mikhailov and Valchev 2011-12, there has been studied the auxiliary linear problem

$$L_{S_1}\psi = (\mathrm{i}\partial_x + \lambda S_1)\psi = 0, \quad S_1 = \left(\begin{array}{ccc} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{array} \right),$$

where $u(x)$, $v(x)$ have the same properties as above. We call this system GMV system. The GMV system arises when one looks for integrable system having a Lax representation $[L, A] = 0$ with $L = \mathrm{i}\partial_x + \lambda S(x)$, where $S(x) \in \mathrm{sl}\,(3, \mathbb{C})$ and L, A are subject to Mikhailov-type reduction requirements. In GMV case the reduction group is generated by two elements g_0 and g_1 acting on the fundamental solutions of the GMV system as :

$$g_0(\psi)(x, \lambda) = \left[\psi(x, \lambda^*)^\dagger\right]^{-1}$$

$$g_1(\psi)(x, \lambda) = H_1\psi(x, -\lambda)H_1, \quad H_1 = \mathrm{diag}\,(-1, 1, 1)$$

Since $g_0 g_1 = g_1 g_0$ and $g_0^2 = g_1^2 = \mathrm{id}$ we see that the reduction group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The reduction group leaves the set of the fundamental solutions invariant and one can prove that this forces the form of S_1 in the GMV system. The normalizing relation is of different nature, in fact one can prove that it ensures that S_1 has constant eigenvalues $+1, 0, -1$, so that S_1 in the orbit of $J_0 = \mathrm{diag}\,(1, 0, -1)$ with respect to the adjoint action of the group $SU(3)$. Thus one is able to show that that the GMV system is gauge-equivalent to a Generalized Zakharov-Shabat system (GZS) on $\mathrm{sl}\,(3, \mathbb{C})$ and as a consequence it has nice spectral properties, identical to that of the GZS. This could be used either to develop its spectral properties and the properties of the so-called Recursion Operators independently (but in analogy with GZS), as has been done in the works we cited, or to develop them using the gauge covariant theory of the Recursion Operators (Gerdjikov,Vilasi, Yanovski 2008 monograph book) as has been done in Yanovski 2011-12, 2014.

The linear problem we shall consider has been introduced together with the GMV and it is in a sense a generalization of GMV problem. Indeed, assume that one wants bigger Mikhailov reduction group, generated this time by the following three elements

$$g_0(\psi)(x,\lambda) = \left[\psi(x,\lambda^*)^\dagger\right]^{-1}$$
$$g_1(\psi)(x,\lambda) = H_1\psi(x,-\lambda)H_1, \quad H_1 = \mathrm{diag}\,(-1,1,1)$$
$$g_2(\psi)(x,\lambda) = H_2\psi(x,\frac{1}{\lambda})H_2, \quad H_2 = \mathrm{diag}\,(1,-1,1).$$

Since the elements g_i, $i = 0,1,2$ commute and $g_i^2 = \mathrm{id}$ the reduction group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Naturally, L_{S_1} cannot admit such such reduction group for which rational dependence on λ is needed. This led to considering the linear problem

$$L_{S_{\pm 1}}\psi = (\mathrm{i}\partial_x + \lambda S_1 + \lambda^{-1} S_{-1})\psi = 0$$

subject to reduction generated by g_0, g_1, g_2.

The invariance under the above reductions force the matrix $S_L(\lambda) = \lambda S_1 + \lambda^{-1} S_{-1}$ to obey:

$$(S_L(\lambda^*))^\dagger = S_L(\lambda), H_1 S_L(-\lambda) H_1 = S_L(\lambda), H_2 S_L(\lambda^{-1}) H_2 = S_L(\lambda)$$

so one has that $S_{-1} = H_2 S_1 H_2$ and taking into account the form of S_1 we get

$$S_{-1} = \begin{pmatrix} 0 & -u & v \\ -u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}.$$

Thus $L_{S_{\pm 1}}$ becomes exactly the linear problem we started with. Gerdjikov, Mikhailov and Valchev 2011-12 considered the question of the Recursion Operators for RGMV and made the first steps into the study of its spectral properties. They made also some important observations how to construct the fundamental solutions analytic in λ (FAS) but limited their scope to the degenerate cases when either u_0 or v_0 is equal to zero. We shall consider now the general position boundary conditions for which both $u_0, v_0 \neq 0$.

FAS asymptotic

In order to write down the integral equations that will give us the fundamental analytic solutions (FAS) we need to know about their asymptotic behavior when $x \to \pm\infty$. We expect that when $x \to \pm\infty$ the solutions of $L_{S_{\pm 1}}\psi = 0$ will behave as $(\exp \mathrm{i}J(\lambda)x)A$ where $A = A(\lambda)$ is a matrix that does not depend on x and $J(\lambda) = S(\lambda)|_{u=u_0, v=v_0}$. It is not hard to find that $J(\lambda)$ has eigenvalues

$$\mu_0 = 0, \quad \mu_\pm = \pm\sqrt{2(|v_0|^2 - |u_0|^2) + (\lambda^2 + \lambda^{-2})}.$$

Since $|u_0|^2 + |v_0|^2 = 1$ one can also cast μ in the following equivalent forms:

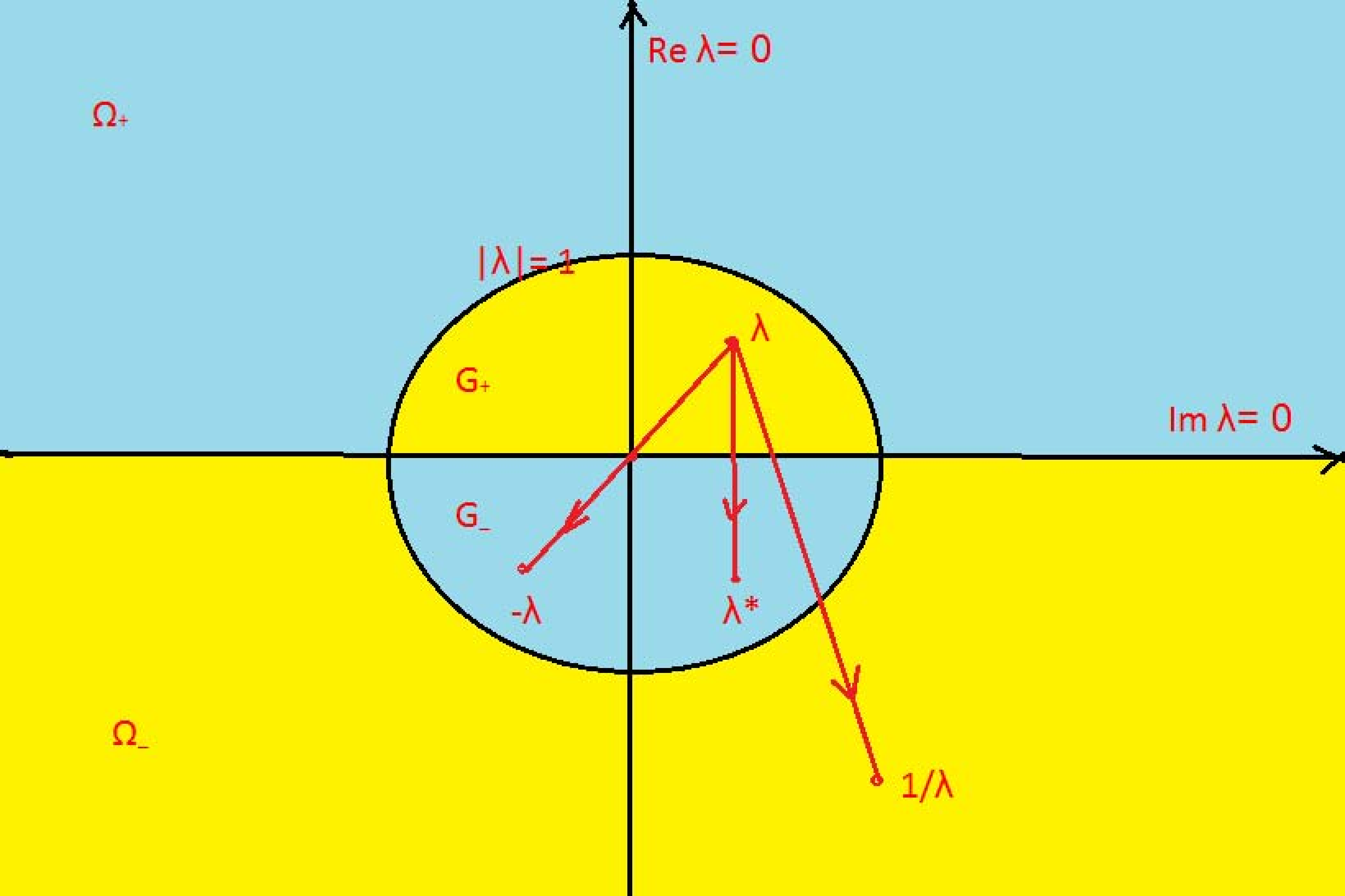
$$\mu = \pm\sqrt{4|v_0|^2 + (\lambda - \lambda^{-1})^2} = \pm\sqrt{-4|u_0|^2 + (\lambda + \lambda^{-1})^2}.$$

Hence $J(\lambda)$ is diagonalizable and there is a constant matrix C (depending of course on u_0 and v_0 and λ) such that

$$C^{-1}J(\lambda)C = \mu(\lambda)\mathrm{diag}\,(1, 0, -1) = \mu(\lambda)J_0.$$

Denote $r(\lambda) = 2(|v_0|^2 - |u_0|^2) + (\lambda^2 + \lambda^{-2})$. Then μ is a square root of $r(\lambda)$ which of course has two branches. In the degenerate cases $r(\lambda)$ becomes a square of an analytic functions having simple poles at $\lambda = 0$ and $\lambda = \infty$ (In fact $(\lambda + \lambda^{-1})^2$ when $u_0 = 0$ and $(\lambda - \lambda^{-1})^2$ when $v_0 = 0$) so the two branches are $\pm(\lambda + \lambda^{-1})$ when $u_0 = 0$ and $\pm(\lambda - \lambda^{-1})$ when $v_0 = 0$.

In case both u_0 and v_0 are different from zero the situation is not so trivial. One of the ways to describe the branches will be to cut the plane into simply connected regions such that in each of them the function $r(\lambda)$ does not have zeros, then in each of them there will be exactly two branches of the square root of $r(\lambda)$. Since all the zeros of $r(\lambda)$ lie on the unit circle (we show this below).It is natural to introduce the four regions $G_\pm$, $\Omega_\pm$ obtained cutting $\mathbb{C}$ by the circle $\mathbb{S}^1$ and the real line $\mathbb{R}$ as seen on the picture.



Re λ= 0
Ω+
|λ|= 1
λ
G+
Im λ= 0
G_
-λ
λ*
Ω_
1/λ

On each them one can define branches of the logarithm of $r(\lambda)$ and hence the branches of the square root. There is however a better way to investigate $\mu(\lambda)$. The function $r(\lambda)$ is meromorphic on the extended plane (Riemann sphere $\mathbb{P}^1$) and at $\lambda = 0$ and $\lambda = \infty$ has a poles of order 2. It has simple zeros at the four points $z_1 = |u_0| + \mathrm{i}|v_0|$, $z_2 = -|u_0| + \mathrm{i}|v_0|$, $z_3 = -|u_0| - \mathrm{i}|v_0|$, $z_2 = |u_0| - \mathrm{i}|v_0|$. They degenerate into two points in case either u_0 or v_0 equals zero. (Into ± 1 in case $v_0 = 0$ and into $\pm\mathrm{i}$ in case $u_0 = 0$). All the zeros lie on the unit circle $\mathbb{S}^1 = \{\lambda : |\lambda| = 1\}$. Let us also remark that since the function $r(\lambda)$ is invariant under the involutions mapping the Riemann sphere into itself:

$$\lambda \mapsto \lambda^*, \quad \lambda \mapsto -\lambda, \quad \lambda \mapsto \lambda^{-1}. \tag{1}$$

The set of zeros is also invariant under these involutions which of course can be checked also directly.

For the analytic continuation of $t\sqrt{r(\lambda)}$ we first remark that $r(\lambda)$ could be written into the form

$$r(\lambda) = \lambda^{-2}(\lambda - z_1)(\lambda - z_2)(\lambda - z_3)(\lambda - z_4).$$

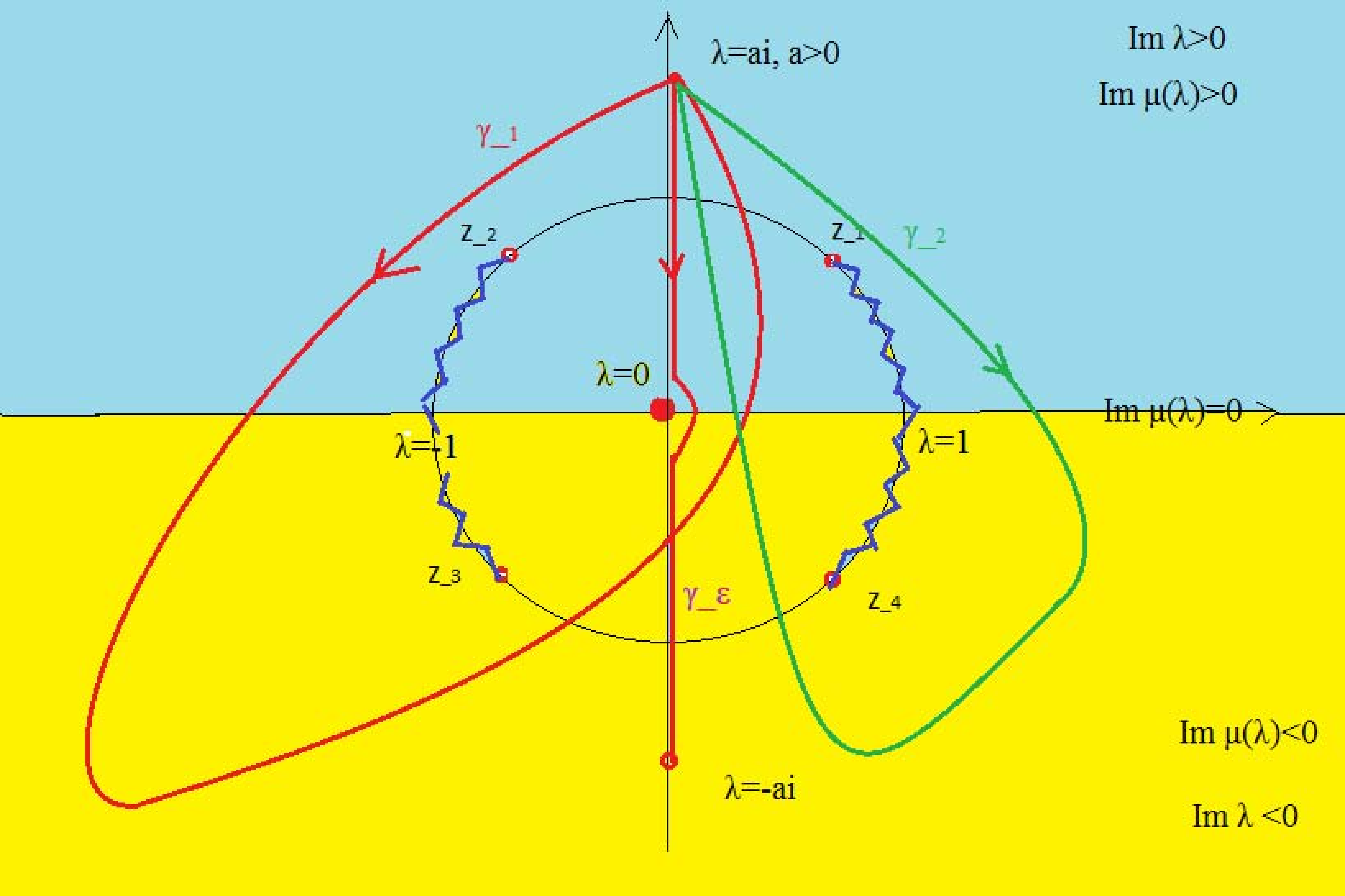
Then one can apply to $r(\lambda)$ the standard technique for analytic continuation of a germ of the square root. For this consider the closed arcs $a_{\pm}$ (see the picture) and let us cut the complex plane along them. Let G_0 be the punctured plane from which are eliminated the cuts. Further, consider

$$\frac{r'(\lambda)}{r(\lambda)} = -\frac{2}{\lambda} + \sum_{k=1}^{4} \frac{1}{\lambda - z_k}.$$

The region G_0 we had in the above is of course not simply connected but as it is readily checked, the integral

$$\Delta_{\gamma} \arg(r(\lambda)) = \frac{1}{\mathrm{i}} \int_{\gamma} \frac{r'(\lambda)}{r(\lambda)} d\lambda$$

over arbitrary closed, partially smooth curve γ is equal to $\pm 4n\pi$ where $n = 0, 1, 2$.



Im λ>0
Im μ(λ)>0
λ=ai, a>0
γ_1
Z_2
Z_1
γ_2
λ=0
Im μ(λ)=0
λ=-1
λ=1
Z_3
Z_4
γ_ε
λ=-ai
Im μ(λ)<0
Im λ <0

Therefore, $\frac{1}{2}\Delta_\gamma \arg(r(\lambda))$ along arbitrary closed curve is a multiple of 2π so in G_0 one can define an analytic continuation of the square root of $r(\lambda)$. We Take as $\mu(\lambda)$ the branch that for $\lambda = \mathrm{i}$ equals $2\mathrm{i}|u_0|$, the other will be $-\mu(\lambda)$.

We shall always take as μ the branch for which $\mathrm{Im}(\mu(\mathrm{i}))$ is positive and in the degenerate case $u_0 = 0$ we shall take the branch for which $\mathrm{Im}\mu(a\mathrm{i})$ positive for $a\mathrm{i}$, a-real $a > 1$.

Next we investigate the sign of $\mathrm{Im}(\mu(\lambda))$. For example, the value of μ for $\lambda = -\mathrm{i}$ is obtained integrating along the curve γ_ϵ. Due the symmetries we are able to calculate the integral and get $\Delta_{\gamma_\epsilon}\frac{1}{2}\arg(r(\lambda)) = -\pi$ so $\mu(-\mathrm{i}) = -2\mathrm{i}|u_0|^2$. There is a simpler way to get what we need and that is to investigate μ^2. We get $\mathrm{Im}(\mu)(\lambda) > 0$ for $\lambda \in G_0$ in the upper half-plane and $\mathrm{Im}(\mu)(\lambda) < 0$ for $\lambda \in G_0$ in the lower half-plane. On the real line (except $\lambda = 0$) we have that $\mathrm{Im}(\mu) = 0$, $\mu \neq 0$. (The function $\mathrm{Im}(\mu)$ (but not μ since $\mathrm{Re}(\mu)$ has a jump) could be extended setting it equal to zero to all points of the cuts and μ could be extended setting it equal to zero at $\lambda = z_i$, $i = 1, 2, 3, 4$.

The function μ is meromorphic in $G_0 \cup \{\infty\}$ and has simple poles at $\lambda = 0$, $\lambda = \infty$. Next, the symmetry properties of $r(\lambda)$ lead to symmetry properties for $\mu(\lambda)$. The first one follows from the fact that if we expand μ in Taylor or Laurent series the coefficients in these expansions will be real. The two other ones follow taking into account that the maps

$$\lambda \mapsto -\lambda, \quad \lambda \mapsto \lambda^{-1}, \quad \lambda \mapsto \lambda^*$$

interchange the upper and lower half-planes, that in each connected open set $\sqrt{r(\lambda)}$ has exactly two branches μ and $-\mu$, and that we know the sign of $\mathrm{Im}(\mu(\lambda))$ is same as the sign of $\mathrm{Im}(\lambda)$. Thus we obtain:

$$\mu^*(\lambda^*) = \mu(\lambda), \quad \mu(\lambda^{-1}) = -\mu(\lambda), \quad \mu(-\lambda) = -\mu(\lambda).$$

All the above is needed for the construction of the fundamental analytic solutions (FAS) of $L_{S_{\pm 1}}$ since it will become important to know the regions in which $\mathrm{Im}(\mu(\lambda))$ is positive (negative). Is also important to know what branch we shall use because on this choice depends the matrix $C = C(\lambda, \mu(\lambda))$ that diagonalizes $J(\lambda)$. We have

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ a(\lambda) u_0^* & -\sqrt{2} b(\lambda) v_0 & -a(\lambda) u_0^* \\ b(\lambda) v_0^* & \sqrt{2} a(\lambda) u_0 & b(\lambda) v_0^* \end{pmatrix}.$$

where $a(\lambda) = \mu^{-1}(\lambda - \lambda^{-1})$ and $b(\lambda) = \mu^{-1}(\lambda + \lambda^{-1})$. We have that $C^{-1} J(\lambda) C = \mu J_0 = \mathrm{diag}\,(\mu, 0, -\mu)$. The matrix $C(\lambda, \mu(\lambda))$ is not unique, we have chosen it to be unitary for real λ since in this case $J(\lambda)$ is Hermitian. Changing μ to $-\mu$, that is passing from $C(\lambda, \mu(\lambda))$ to $C(\lambda, -\mu(\lambda))$ is equivalent to multiplying $C(\lambda, \mu(\lambda))$ to the left by $\mathrm{diag}\,(1, -1, -1) = -H_1$. We shall write $C_+(\lambda) = C(\lambda, \mu(\lambda))$ and $C_-(\lambda) = C(\lambda, -\mu(\lambda))$. Thus $C_- = -H_1 C_+$.

Let us assume that ϕ is a solution to the RGMV system:

$$L_{S_{\pm 1}}\phi = (\mathrm{i}\partial_x + \lambda S_1(x) + \lambda^{-1} S_{-1}(x))\phi = 0.$$

As pointed out in Gerdjikov,Mikhailov,Valchev 2012 in order to investigate the fundamental solutions of RGMV it is useful to introduce the functions

$$\Phi_{\pm}(x,\lambda) = C_{\pm}^{-1}(\lambda)\phi(x,\lambda)\exp\left(\mp\mathrm{i}\mu(\lambda)J_0 x\right)$$

which satisfy the equation

$$\mathrm{i}\partial_x\Phi_{\pm} + S_{\pm}[\lambda]\Phi \mp \mu(\lambda)\Phi_{\pm}J_0 = 0.$$

In it for the sake of brevity we have put

$$S_{\pm}[\lambda, x] = \lambda(C_{\pm}^{-1}S_1 C_{\pm}) + \lambda^{-1}(C_{\pm}^{-1}S_{-1}C_{\pm})$$

We shall use also $S_{\pm}[\lambda, x] = S_{\pm}[\lambda]$ and shall call the above system(s) of DE modified RGMV system (s). Conversely, if $\Phi_{+}(x,\lambda)$ and $\Phi_{+}(x,\lambda)$ satisfy the corresponding modified RGMV systems then both functions

$$\psi_{\pm} = C_{\pm}\Phi_{\pm}\exp\left(\pm\mathrm{i}\mu(\lambda)J_0\right)$$

satisfy RGMV.

FAS. Integral equations

Since the number of subscripts and superscripts starts to grow quickly we shall concentrate first on the functions Φ_+ and shall not write the subscript $+$. Let us try to find functions $\zeta^{p,m}$ that satisfy the above equation and in addition have asymptotic $\lim_{x\to-\infty}\zeta^m(x) = \mathbf{1}$, $\lim_{x\to+\infty}\zeta^p(x) = \mathbf{1}$. Such solutions can be constructed through a procedure that applied for the GZS system and its generalization - the Caudrey-Beals-Coifman (CBC) system, Beals, Coifman 1994.
One needs to consider two separate cases: a) $\mathrm{Im}(\mu(\lambda)) > 0$ and b) $\mathrm{Im}(\mu(\lambda)) < 0$. The systems of integral equations will be written for the components of the 3×3 matrix functions $\zeta^n(x,\lambda)$ and $\zeta^p(x,\lambda)$.

Case (a). $\mathrm{Im}(\mu(\lambda)) > 0$, fixed asymptotic at $-\infty$. (For boundary conditions in general position this means that λ belongs to the upper half-plane without the cuts). Solutions are denoted by $\zeta^n(x,\lambda)$ and the system of the integral equations runs as:

$$\zeta^n_{jk}(x,\lambda) = \delta_{jk} +$$

$$\mathrm{i}\sum_{s=1}^{3}\int_{-\infty}^{x} \mathrm{d}y (S_+[\lambda](y) - \mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}(\mu_{jk})(x-y)\right]$$

when $j \leq k$ and when $j > k$

$$\zeta^n_{jk}(x,\lambda) =$$

$$\mathrm{i}\sum_{s=1}^{3}\int_{+\infty}^{x} \mathrm{d}y (S_+[\lambda](y) - \mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}\mu_{jk}(x-y)\right].$$

Case (b). $\mathrm{Im}(\mu(\lambda)) < 0$, fixed asymptotic at $-\infty$. (For boundary conditions in general position this means that λ belongs to the lower half-plane without the cuts). Then we must consider the following system:

$$\zeta^n_{jk}(x,\lambda) = \delta_{jk} +$$

$$\mathrm{i}\sum_{s=1}^{3}\int_{-\infty}^{x} \mathrm{d}y (S_+[\lambda](y) - \mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}\mu_{jk}(x-y)\right]$$

when $j \geq k$ and when $j < k$

$$\zeta^n_{jk}(x,\lambda) =$$

$$\mathrm{i}\sum_{s=1}^{3}\int_{+\infty}^{x} \mathrm{d}y (S_+[\lambda](y) - \mu(\lambda)J_0)_{js}\zeta^n_{sk}(y,\lambda)\exp\left[\mathrm{i}\mu_{jk}(x-y)\right].$$

In the above $\mu_{kk} = 0$, $\mu_{12} = -\mu_{21} = \mu$, $\mu_{13} = -\mu_{31} = 2\mu$, $\mu_{23} = -\mu_{32} = \mu$. One sees that for $i < j$ we have $\mathrm{Im}(\mu_{ij}) > 0$ in the upper half-plane and $\mathrm{Im}(\mu_{ij}) < 0$ in the lower half-plane while for $i > j$ we have $\mathrm{Im}(\mu_{ij}) < 0$ in the upper half-plane and $\mathrm{Im}(\mu_{ij}) > 0$ in the lower half-plane. Thus for $j \neq k$ in the integrands are always present falling exponents ensuring that the kernels of the above integral operators fall exponentially when $x \to \pm\infty$. It is expected that provided the function $(S_+[\lambda] - \mu(\lambda)J_0)$ has small $L^1(\mathbb{R})$ norm the above equations have solutions $\zeta^{n,+}(x, \lambda)$ (for λ in the upper half-plane without the cuts) and $\zeta^{n,-}(x, \lambda)$ (for λ in the lower half-plane without the cuts). We shall omit the superscripts $+$ $(-)$ assuming that when λ is in the upper half-plane we have $\zeta^{n,+}(x, \lambda)$ and when it is in the lower half-plane we have $\zeta^{n,-}(x, \lambda)$. We shall preserve the superscripts only when it is necessary to avoid ambiguity, for example, when λ is real and we must know whether we have extension of the corresponding functions from above (below).

For the functions ζ^p analogous systems of integral equations can be constructed, we shall not write them here. Finally, for the degenerate cases the constructions of the solutions $\zeta^{n,p}$ do not differ from the one we had in the above, simply the analyticity regions change and only in the case $u_0 = 0$. Let us not forget also that our functions have subscript $+$ or $-$ depending on the choice of the branch for μ. In all the cases it is readily checked that if $\zeta^n(x,\lambda)$, $\zeta^p(x,\lambda)$ are bounded and satisfy the above systems of integral equations then they are solutions to the modified RGMV system:

$$\mathrm{i}\partial_x\Phi_\pm + S_\pm[\lambda]\Phi \mp \mu(\lambda)\Phi_\pm J_0 = 0.$$

Our ultimate intention is to establish a theorem, that is an analog of a theorem one has for the CBC system. Thus we expect to establish that:

- For potentials that go fast to their limit values (this words should be given precise meaning) the integral equations for $\zeta^n_\pm(x,\lambda), \zeta^p_\pm(x,\lambda)$ have unique solutions which satisfy

$$\lim_{x\to-\infty} \zeta^n_\pm(x,\lambda) = \mathbf{1}, \qquad \lim_{x\to+\infty} \zeta^p_\pm(x,\lambda) = \mathbf{1}.$$

- These functions are analytic in the regions where $\mathrm{Im}(\mu(\lambda)) \neq 0$ and could be extended by continuity to their boundaries.
- For a class of potentials that do not go fast to their limit values, and that should be further specified, the fundamental solutions $\zeta^n_\pm(x,\lambda), \zeta^p_\pm(x,\lambda)$ possibly do not exist in a finite number of points where they have pole type singularities – the discrete spectrum of the RGMV linear problem.
- The Mikhailov reduction symmetries g_1, g_2, g_3 should result in symmetries of the solutions $\zeta^n_\pm(x,\lambda), \zeta^p_\pm(x,\lambda)$.

The present article is a first step in the above program. We start by considering the question of uniqueness.

Proposition

Suppose for given potentials $u(x), v(x)$ and $\mathrm{Im}(\lambda) \neq 0$ the bounded fundamental solutions $\zeta^n_{\pm}(x, \lambda), \zeta^p_{\pm}(x, \lambda)$ exist. Then they are unique.

The proof is quite similar to the proof given in the case of the GZS (CBC) system. Also standard is the next step – to establish relation between the solutions $\zeta^{n,+}(x, \lambda)$ and $\zeta^{p,+}(x, \lambda)$. It is not hard to prove that there exists diagonal matrices $D^{\pm}(\lambda)$ (that with '+' analytic in the upper half-plane without the cuts and the other with '-', analytic in the lower half-plane without the cuts) such that

$$\begin{aligned} \zeta^{n,+}(x,\lambda)D^{+}(\lambda) &= \zeta^{p,+}(x,\lambda), \qquad \mathrm{Im}(\lambda) > 0 \\ \zeta^{n,-}(x,\lambda)D^{-}(\lambda) &= \zeta^{p,-}(x,\lambda), \qquad \mathrm{Im}(\lambda) < 0. \end{aligned}$$

Since for example $\lim_{x\to+\infty} \zeta^{n,+}(x, \lambda) = D^{+}(\lambda)$ one can recover $\zeta^p(x, \lambda$ from $\zeta^n(x, \lambda)$ and so it is enough to consider only solutions of the type ζ^n.

It even more interesting however to consider the relation between $\zeta^{+}(x,\lambda)$ and $\zeta^{-}(x,\lambda)$. Of course we are speaking here about λ such that $\mathrm{Im}(\mu(\lambda))=0$ (which in case of general position boundary conditions means that $\lambda\in\mathbb{R}\setminus\{-1,0,1\}$. In this case the exponential factors are always bounded and we have

$$\zeta^{+}(x,\lambda)=\zeta^{-}(x,\lambda)e^{-\mathrm{i}\mu(\lambda)xJ_0}G(\lambda)e^{\mathrm{i}\mu(\lambda)xJ_0}.$$

for some non-degenerate matrix $G(\lambda)$ defined in $\mathbb{R}\setminus\{-1,0,1\}$. The above relation is essential for finding appropriate scattering data and possible formulation of Inverse Scattering Method for the RGMV system. Let us also remark that the behavior of this relation at the points $\lambda=0,\pm$ should be additionally investigated.

FAS. Existence, 'small' potentials

For definiteness we shall assume $\mathrm{Im}(\lambda) > 0$ (without the cuts). Let us introduce the following norms. If $R(x)$ is a 3×3 matrix whose entries are complex functions on the line we set

$$||R||_1 = \max_{1 \leq i,j \leq 3} \int_{-\infty}^{+\infty} |R_{ij}(x)| \mathrm{d}x.$$

if its entries belong to $L^1(\mathbb{R})$ and for $R(x)$ whose entries are bounded we define

$$||R||_\infty = \sup_{x \in \mathbb{R}; 1 \leq i,j \leq 3} |R_{ij}|(x).$$

The space with the norm $|| \; ||_1$ we denote by $L^1(\mathbb{R}; 9)$ and the space with the norm $|| \; ||_\infty$ by $L^\infty(\mathbb{R}; 9)$. Both these spaces are Banach spaces. Then we have

Proposition

Suppose Ω is an open subset in the upper half-plane (without the cuts) with compact closure that do not contain $\lambda = 0$. Suppose that for any $\lambda \in \bar{\Omega}$ the the function $Q(x, \lambda) = (S_+[\lambda, x] - \mu(\lambda)J_0)$ belongs to $L^1(\mathbb{R}, 9)$ and $||Q(x, \lambda)||_1 < 1$ (small potentials on Ω). Then for $\lambda \in \Omega$ there exists unique solution $\zeta^+(x, \lambda)$ of the integral equations with the following properties:

1. *The solution $\zeta^+(x, \lambda)$ together with its λ-derivatives allows continuous extension to the closure $\bar{\Omega}$ of Ω.*
2. *The solution $\zeta^+(x, \lambda)$ and its inverse obey the estimates*

$$||\zeta^+||_\infty < (1 - \alpha)^{-1}, \qquad ||(\zeta^+)^{-1}||_\infty < (1 - \alpha)^{-1}$$

where $\alpha = \max_{\lambda \in \bar{\Omega}} ||Q(x, \lambda)||_1 < 1$.

The proof is not hard, its idea is to write the integral equation into a form for which the Banach fixed point theorem could be applied.

FAS. The effect of the symmetries

Let us now find how the symmetries we had for our linear problem RGMV affect the solutions we introduced. In this subsection we shall assume that the fundamental solutions $\zeta_{\pm}(x, \lambda)$ exist.

Lemma

Suppose we have the general position boundary conditions. Then the matrices $C_{+}(\lambda)$, $C_{-}(\lambda) = -H_1 C_{+}(\lambda)$ satisfy the relations:

$$C_{-}(\lambda) = -H_1 C_{+}(\lambda), \quad [(C_{\pm}(\lambda^*)^{\dagger}]^{-1} = C_{\pm}(\lambda, \mu(\lambda))$$
$$H_2 C_{\pm}(\lambda^{-1}) H_2 = C_{\mp}(\lambda), \quad C_{\pm}(-\lambda) = C_{\pm}(\lambda)$$
$$C_{\pm}(-\lambda) = C_{\pm}(\lambda).$$

So there are relations showing how the symmetries affect $S_{\pm}[\lambda, x]$ and finally the solutions ζ:

Proposition

In the case of general position boundary conditions the solutions $\zeta(x,\lambda)$ have the following properties:

$$[(\zeta_{\pm}(x,\lambda^*)^{\dagger}]^{-1} = \zeta_{\pm}(x,\lambda)\ H_2\zeta_{\pm}(x,\lambda^{-1})H_2 = \zeta_{\mp}(x,\lambda)$$
$$\zeta_{\pm}(x,-\lambda) = \zeta_{\mp}(x,\lambda).$$

Finally, in terms of the solutions

$$\chi_{\pm}(x,\lambda) = C_{\pm}(\lambda)\zeta_{\pm}(x,\lambda)\exp\left(\pm\mathrm{i}\mu(\lambda)xJ_0\right)$$

of RGMV the above symmetries take the form:

Proposition

In the case of general position boundary conditions the solutions $\chi(x,\lambda)$ have the following properties:

$$[(\chi_{\pm}(x,\lambda^*)^{\dagger}]^{-1} = \chi_{\pm}(x,\lambda),\ \ H_2\chi_{\pm}(x,\lambda^{-1})H_2 = \chi_{\mp}(x,\lambda)$$
$$H_1\chi_{\pm}(x,-\lambda)H_1 = \chi_{\mp}(x,\lambda)(-H_1).$$

Conclusions

We have investigated the problem of fundamental analytic solutions (FAS) for the operator $L_{S_{\pm 1}}$ in case of boundary conditions in general position. We established the uniqueness and the symmetry property for these solutions. Also, we have established some results about the existence of FAS which however should be considerably improved in order to prove that FAS exist for a reasonable class of potentials. This is a trend we are going to follow in the next future. The big goal is to establish completeness relations constructed through the FAS of $L_{S_{\pm 1}}$. This will permit to extend the known theory of expansions over the so-called adjoint solutions which is basic for the Recursion Operators approach to soliton equations to the equations solvable though the auxiliary linear problem $L_{S_{\pm 1}}\psi = 0$.

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Thank you for your attention!