Some results on Frenet ruled surfaces along the evolute-involute curves, based on normal vector fields in E^3.

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Abstract

We consider eight special Frenet ruled surfaces along to the involute-evolute curves, α^* and α respectively, with curvature $k_1 \neq 0$. First we find the excplit equation of Frenet ruled surfaces along the involute curves in terms of the Frenet apparatus of evolute curve α. Further normal vector fields of these Frenet ruled surfaces be calculated too.
In this paper we give all results for sixteen positions of Normal vector fields of four Frenet ruled surfaces in terms of Frenet apparatus of evolute curve α. These results also give us the positions of eight special Frenet ruled surfaces along to the involute-evolute curves, based on their normal vectors, in terms of curvatures of evolute curve α. We can give the answer of the question that in which condition we can produce orthogonal surfaces or surfaces with constant angle .

Introduction and Preliminaries

Deriving curves based on the other curves is a subject in geometry. Involute-evolute curves, Bertrand curves are this kind of curves. By using the similiar method we produce a new ruled surface based on the other ruled surface. The Involutive $B - scrolls$ are defined in [1]. $\tilde{D} - scroll$, which is known as the rectifying developable surface, of any curve α and the *involute* $\tilde{D} - scroll$ of the curve α are already defined, in Euclidean $3 - space$. Also the differential geometric elements of the *involute* $\tilde{D}$ scroll are examined in [2]. In this paper we consider the following four special ruled surfaces associated to a space curve α with $k_1 \neq 0$. They are called as Frenet ruled surface, cause of their generators are the Frenet vector fields of a curve.

Introduction and Preliminaries

It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves.

Introduction and Preliminaries

Let Frenet vector fields be $V_1(s), V_2(s), V_3(s)$ of α and let the first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. The quantities $\{V_1, V_2, V_3, \tilde{D}, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves.

Introduction and Preliminaries

For any unit speed curve α,in terms of the Frenet-Serret apparatus, the Darboux vector D can be expressed as $D(s) = k_2(s)V_1(s) + k_1(s)V_3(s)$. Let a vector field be $\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s)$ along $\alpha(s)$ under the condition that $k_1(s) \neq 0$ and it is called the modified Darboux vector field of α [4]. The Darboux vector field of α and it has the following symmetrical properties [5]:
$D \times V_1 = \dot{V}_1; D \times V_2 = \dot{V}_2; D \times V_3 = \dot{V}_3$. Also it is trivial that $\tilde{D}(s)' = \left(\frac{k_2}{k_1}\right)' V_1$.

Introduction and Preliminaries

The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where curvature functions are defined by $k_1 = k_1(s) = \|V_1(s)\|$ and $k_2(s) = -\left\langle V_2, \dot{V}_3 \right\rangle$.

The Involute Curve

The involute of a given curve is a well-known concept in Euclidean $3-space$. We can say that; evolute and involute is a method of deriving a new curve based on a given curve. The involute of the curve is called some times *the evolvent*. Involvents play a part in the construction of gears. The evolute is the locus of the centers of tangent circles of the given planar curve [7].

The Involute Curve

Let α and α^* be the curves in Euclidean $3-space$. The tangent lines to a curve α generate a surface called the tangent surface of α. If the curve α^* which lies on the tangent surface of α and intersects the tangent lines orthogonally is called an involute of α [5]. If a curve α^* is an involute of α, then by definition α is an evolute of α^*. The quantities $\left\{V_1, V_2, V_3, \tilde{D}, k_1, k_2\right\}$ and $\left\{V_1^*, V_2^*, V_3^*, \tilde{D}^*, k_1^*, k_2^*\right\}$ are collectively Frenet-Serret apparatus of the *evolute* α and the *involute* α^*, respectively.

The Involute Curve

In the Euclidean $3-space$ $\mathbf{E}^3$, α , $\alpha^* \subset \mathbf{E}^3$, α and α^* are the arclengthed curves with the arcparametres s and s^*, respectively.

$$\alpha^*(s) = \alpha(s) + \lambda V_1(s), \ \lambda = \rho - s$$

is the equation of involute of the curve α. Then we have $\langle V_1^*, V_1 \rangle = 0$, $V_2 = V_1^*$. (For more detail see in [6], [5])

The Involute Curve

The following result shows that we can write the Frenet apparatus of the involute curve based on the Frenet apparatus of the evolute curve (For more detail see in [6], [5]).

The Involute Curve

Theorem

The Frenet vectors of the involute α^*, based on the its evolute curve α are

$$V_1^* = V_2,$$
$$V_2^* = \frac{-k_1 V_1 + k_2 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}}$$
$$V_3^* = \frac{k_2 V_1 + k_1 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}}$$
$$\tilde{D}^* = \frac{k_2}{\sqrt{k_1^2 + k_2^2}} V_1 - \frac{k_1' k_2 - k_1 k_2'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} V_2 + \frac{k_1 V_3}{\sqrt{k_1^2 + k_2^2}}.$$

The Involute Curve

Theorem

The first and the second curvatures of involute α^* are

$$k_1^* = \frac{\sqrt{k_1^2 + k_2^2}}{(\rho - s)k_1}, (\rho - s)k_1 > 0, \ k_1 \neq 0.$$

$$k_2^* = \frac{k_2^{'} k_1 - k_1^{'} k_2}{\lambda k_1 \left(k_1^2 + k_2^2\right)} = \frac{-k_2^2 \left(\frac{k_1}{k_2}\right)^{'}}{\lambda k_1 \left(k_1^2 + k_2^2\right)}, \lambda = \rho - s,$$

respectively.

The Involute Curve

Theorem

The product of Frenet vector fields of the involute α^*, and its evolute curve α has the following matrix form;

$$[V]\,[V^*]^T = \frac{1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}}\begin{bmatrix} 0 & -k_1 & k_2 \\ \left(k_1^2+k_2^2\right)^{\frac{1}{2}} & 0 & 0 \\ 0 & k_2 & k_1 \end{bmatrix}.$$

The Involute Curve

Proof.

Since

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{bmatrix} V_1^* & V_2^* & V_3^* \end{bmatrix} = \begin{bmatrix} \langle V_1, V_1^* \rangle & \langle V_1, V_2^* \rangle & \langle V_1, V_3^* \rangle \\ \langle V_2, V_1^* \rangle & \langle V_2, V_2^* \rangle & \langle V_2, V_3^* \rangle \\ \langle V_3, V_1^* \rangle & \langle V_3, V_2^* \rangle & \langle V_3, V_3^* \rangle \end{bmatrix}$$

□

Proof.

and

$$\langle V_1, V_3^* \rangle = \frac{k_2}{(k_1^2 + k_2^2)^{\frac{1}{2}}}$$

$$\langle V_3, V_3^* \rangle = \frac{k_1}{(k_1^2 + k_2^2)^{\frac{1}{2}}}$$

$$\langle V_2, V_2^* \rangle = 0; \qquad \langle V_1, V_1^* \rangle = 0$$

$$\langle V_2, V_3^* \rangle = 0; \qquad \langle V_3, V_1^* \rangle = 0$$

we get the proof. □

Ruled surface

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3-space$ [3], [2]. Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\left\langle \alpha', v \right\rangle = 0$. To illustrate the current situation, we bring here the famous example of L. K. Graves (see [4]), so called the $B-scroll$. The special ruled surfaces $B-scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves. The Gauss map of B-scrolls has been examined in [3].

Ruled surface

In the Euclidean $3-space$, the ruled surface which is called $involutive\ B-scroll$ (binormal scroll) of the curve α has been alredaydefined in [1]. The involute curve of $\alpha^{*}(s)=\alpha(s)+(\sigma-s)V_{1}(s)$ is the directrix of this surface. The generating space of $B-scroll$ is spaned by binormal subvector V_{3}^{*}. Here $Sp\{V_{1}^{*},V_{2}^{*}\}$ is the osculator plane of the curve β.

Ruled surface

The ruled surface $B - scroll$ is a surface which can be parametrized as $X(s,t) = \alpha(s) + tB(s)$, a “ruled surface” in Lorentzian $3 - space$ $\mathbf{L}^3$ with null directrix curve and null rulings, i.e., $\alpha(s)$ being a null curve and $B(s)$ a null normal vector field along $\alpha(s)$, satisfying $\langle \dot{\alpha}, B \rangle = -1$ [4].

The fundamental forms of the $B - scroll$ with null directrix and Cartan frame in the Minkowskian 3-space is examined in [4]. The properties of the $B - scroll$ are also examined in Euclidean $3 - space$ and $n - space$ and in Lorentzian $3 - space$ and $n - space$ with time-like directrix curve and null rulings (see [1], [2], [3]).

Frenet ruled surfaces

In this subsection Tangent, Normal, Binormal, Darboux ruled surfaces of any curve are collectively named Frenet ruled surfaces. They have the following equations.

Frenet ruled surfaces

Definition

(Tangent ruled surface) In the Euclidean $3-space$, let $\alpha(s)$ be the arclengthed curve. The equation

$$\varphi^{1}\left(s,u_{1}\right)=\alpha\left(s\right)+u_{1}V_{1}\left(s\right)$$

is the parametrization of the ruled surface which is called $V_1 - scroll$ (tangent ruled surface)**.** The directrix of this $V_1 - scroll$ is the curve $\alpha(s)$. The generating space of this $V_1 - scroll$ is spaned by tangent subvector V_1.

Frenet ruled surfaces

Definition

(Normal ruled surface) In the Euclidean $3-space$, let $\alpha(s)$ be the arclengthed curve. The equation

$$\varphi^2\left(s,u_2\right)=\alpha\left(s\right)+u_2V_2\left(s\right)$$

is the parametrization of the ruled surface which is called $V_2-scroll$ (normal ruled surface)**.** The directrix of this $V_2-scroll$ is the curve $\alpha\left(s\right)$. The generating space of this $V_2-scroll$ is spaned by normal subvector V_2.

Frenet ruled surfaces

Definition

(Binormal ruled surface) In the Euclidean $3-space$, let $\alpha(s)$ be the arclengthed curve. The equation

$$\varphi^3 (s, u_3) = \alpha (s) + u_3 V_3 (s)$$

is the parametrization of the ruled surface which is called $V_3 - scroll$ (binormal ruled surface)**.** The directrix of this $V_3 - scroll$ is the curve $\alpha (s)$. The generating space of this $V_3 - scroll$ is spaned by binormal subvector V_3.

Frenet ruled surfaces

Definition

(**Darboux ruled surface**)In the Euclidean $3-space$, let $\alpha(s)$ be the arclengthed curve. The equation $\varphi^4(s, u_4) = \alpha(s) + u_4\tilde{D}(s)$ or

$$\varphi^4(s, u_4) = \alpha(s) + u_4\left(\frac{k_2}{k_1}(s)V_1(s) + V_3(s)\right)$$

is the parametrization of the ruled surface which is called rectifying developable surface of the curve α in [4]. Here, it is referred to as the **Darboux ruled surface** because the generator vector is modified Darboux vector field $\tilde{D}$.

The following theorem gives us a simple matrix for η_1, η_2, η_3, and η_4 which are four normal vector fields of Frenet ruled surfaces.

Frenet ruled surfaces

Theorem

In the Euclidean 3 − space, let $\eta_1, \eta_2, \eta_3,$ *and* η_4 *be the normal vector fields of ruled surfaces* $\varphi^1, \varphi^2, \varphi^3,$ *and* φ^4*, recpectively, along the curve* α*. They can be expressed by the following matrix;*

$$[\eta] = [A]\,[V];$$

$$[\eta] = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a & 0 & b \\ c & d & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Frenet ruled surfaces

Theorem

where
$$a = \frac{-u_2 k_2}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}} \qquad c = \frac{-u_3 k_2}{\sqrt{(u_3 k_2)^2 + 1}}$$
$$b = \frac{(1 - u_2 k_1)}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}} \qquad d = \frac{-1}{\sqrt{(u_3 k_2)^2 + 1}}$$

Frenet ruled surfaces

Proof.

The normal vector fields η_1, η_2, η_3, and η_4 of ruled surfaces $\varphi^1, \varphi^2, \varphi^3$, and φ^4 can be expressed as in the following four equalities

$$\eta_1 = -V_3 \qquad \eta_2 = \frac{-u_2 k_2 V_1 + (1 - u_2 k_1) V_3}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}}$$

$$\eta_3 = \frac{-u_3 k_2 V_1 - V_2}{\sqrt{(u_3 k_2)^2 + 1}} \qquad \eta_4 = -V_2$$

□

Frenet ruled surfaces

Proof.

They can be expressed by the following matrix;

$$[\eta] = \begin{bmatrix} 0 & 0 & -1 \\ \frac{-u_2 k_2}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}} & 0 & \frac{(1 - u_2 k_1)}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}} \\ \frac{-u_3 k_2}{\sqrt{(u_3 k_2)^2 + 1}} & \frac{-1}{\sqrt{(u_3 k_2)^2 + 1}} & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

□

Involutive Frenet ruled surfaces

In this subsection, we give the tangent, normal, binormal, Darboux Frenet ruled surfaces of the *involute* α^* of the *evolute* α have been given as in the following definitions. First we find the excplit equation of Frenet ruled surfaces along the involute curves. Then we write their parametric equations in terms of the Frenet apparatus of the evolute curve α. Thats why they are called " the involutiveTangent, Normal, Binormal, Darboux Frenet ruled surfaces of evolute α as in the following way.

Involutive Frenet ruled surfaces

Definition

(The involutive tangent ruled surface) In the Euclidean $3-space$, let $\alpha(s)$ be an arclengthed curve. The equation $\varphi^{*1}(s, v_1) = \alpha^*(s) + v_1 V_1^*(s)$ is the parametrization of the ruled surface which is called $V_1^* - scroll$ (tangent ruled surface)**.** The directrix of this $V_1^* - scroll$ is the curve $\alpha^*(s)$. The generating space of this $V_1^* - scroll$ is spaned by tangent subvector V_1^*. Also we can write

$$\begin{aligned}\varphi^{*1}(s, v_1) &= \alpha^*(s) + v_1 V_1^*(s) \\ &= \alpha(s) + (\sigma - s) V_1(s) + v_1 V_2(s)\end{aligned}$$

Here we renamed this surface as **the involutive tangent ruled surface of the curve** α, couse of we can write its parametric equation based on the Frenet apparatus of the curve α.

Involutive Frenet ruled surfaces

Definition

(The involutive normal ruled surface) In the Euclidean $3-space$, let $\alpha(s)$ be an arclengthed curve. The equation $\varphi^{*2}(s, v_2) = \alpha^*(s) + v_2 V_2^*(s)$ is the parametrization of the ruled surface which is called $V_2^* - scroll$ (normal ruled surface). The directrix of this $V_2^* - scroll$ is the curve $\alpha^*(s)$. The generating space of this $V_2^* - scroll$ is spaned by normal subvector V_2^*. Also we can write $\varphi^{*2}(s, v_2) = \alpha(s) + (\sigma - s) V_1 + v_2 \left(\frac{-k_1 V_1 + k_2 V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}} \right)$.

Here we renamed this surface as **the involutive normal ruled surface of the curve** α, cause of we can write its parametric equation based on the Frenet apparatus of the curve α.

Involutive Frenet ruled surfaces

Definition

(The involutive binormal ruled surface) In the Euclidean $3-space$, let $\alpha(s)$ be an arclengthed curve. The equation $\varphi^{*3}(s, v_3) = \alpha^*(s) + v_3 V_3^*(s)$ is the parametrization of the ruled surface which is called $V_3^* - scroll$ (binormal ruled surface). The directrix of this $V_3^* - scroll$ is the curve α^* The generating space of this $V_3^* - scroll$ is spaned by binormal subvector V_3^*. Also we can write

$$\varphi^{*3}(s, v_3) = \alpha(s) + (\sigma - s) V_1 + v_3 \left(\frac{k_2 V_1 + k_1 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}} \right).$$

Here we renamed this surface as **the involutive binormal ruled surface of the curve** α, cause of we can write its parametric equation based on the Frenet apparatus of the curve α.

Involutive Frenet ruled surfaces

Definition

(**The involutive Darboux ruled surface**) In the Euclidean $3-space$, let $\alpha(s)$ be an arclengthed curve. The equation $\varphi^{*4}(s, v_4) = \alpha^*(s) + v_4 \tilde{D}^*(s)$ is the parametrization of the ruled surface which is called rectifying developable surface of the curve α in [4]. Also we can write $\varphi^{*4}(s, v_4) =$

$$\alpha(s) + (\sigma - s) V_1 + v_4 \frac{k_2}{\sqrt{k_1^2+k_2^2}} V_1 - v_4 \frac{k_1' k_2 - k_1 k_2'}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} V_2 + v_4 \frac{k_1}{\sqrt{k_1^2+k_2^2}} V_3.$$

Here we renamed this surface as **the involutive Darboux ruled surface of the curve** α, cause of we can write its parametric equation based on the Frenet apparatus of the curve α.

We give more usefull way to express the normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of ruled surfaces $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$ with a matrix with the following theorems.

The normal vector fields of Involutive Frenet ruled surfaces

Theorem

In the Euclidean $3-$ *space , the normal vector fieldS* $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ *of ruled surfaces* $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$ *, recpectively, along the curve involute* α^**, can be expressed by the following matrix*$[\eta^*] = [A^*]\,[V^*]$;

$$[\eta^*] = \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \\ \eta_4^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix},$$

The normal vector fields of Involutive Frenet ruled surfaces

where

$$a^* = \frac{-v_2 k_2^*}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}} \quad c^* = \frac{-v_3 k_2^*}{\sqrt{(v_3 k_2^*)^2 + 1}}$$

$$b^* = \frac{(1 - v_2 k_1^*)}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}} \quad d^* = \frac{-1}{\sqrt{(v_3 k_2^*)^2 + 1}}.$$

The normal vector fields of Involutive Frenet ruled surfaces

Theorem

In the Euclidean 3 − space, the normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ *of ruled surfaces* $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$, *recpectively, can be expressed in terms of the Frenet apparatus of evolute curve* α *by the following matrix;*

$$a^* = \frac{v_2 k_2^2 \left(\frac{k_1}{k_2}\right)'}{\sqrt{v^2\left(-k_2^2\left(\frac{k_1}{k_2}\right)'\right)^2 + \left(k_1^2+k_2^2\right)^2\left(\lambda k_1 - v\sqrt{k_1^2+k_2^2}\right)^2}}$$

$$b^* = \frac{\left(\lambda k_1\left(k_1^2+k_2^2\right) - v_2\left(k_1^2+k_2^2\right)^{\frac{3}{2}}\right)}{\sqrt{v^2\left(-k_2^2\left(\frac{k_1}{k_2}\right)'\right)^2 + \left(k_1^2+k_2^2\right)^2\left(\lambda k_1 - v\sqrt{k_1^2+k_2^2}\right)^2}}$$

The normal vector fields of Involutive Frenet ruled surfaces

$$c^* = \frac{v_3 k_2^2 \left(\frac{k_1}{k_2}\right)'}{\sqrt{v_3^2 \left(-k_2^2 \left(\frac{k_1}{k_2}\right)'\right)^2 + \left(\lambda k_1 \left(k_1^2 + k_2^2\right)\right)^2}}$$

$$d^* = \frac{-\lambda k_1 \left(k_1^2 + k_2^2\right)}{\sqrt{v_3^2 \left(k_2^2 \left(\frac{k_1}{k_2}\right)'\right)^2 + \left(\lambda k_1 \left(k_1^2 + k_2^2\right)\right)^2}}$$

some results

In this section the positions of unit normal vector fields's Frenet ruled surfaces and involutive Frenet ruled surface are examined with the following matrices. We have some results about the condition of the positions of Frenet ruled surfaces and involutive Frenet ruled surface according to their unit normal vector fields.

some results

Theorem

In the Euclidean $3-space$, the position of the unit normal vector field $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of ruled surfaces $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$, recpectively, along the curve involute α^*, can be expressed by the following matrix $[\eta]\,[\eta^*]^T =$

$$\frac{1}{e}\begin{bmatrix} k_1 & -k_1 b^* & -k_2 d^* & k_2 \\ -ak_2 - bk_1 & (ak_2 + bk_1)\, b^* & (-ak_1 + bk_2)\, d^* & ak_1 - bk_2 \\ -ck_2 & eda^* + k_2 cb^* & edc^* - k_1 cd^* & ck_1 \\ 0 & -a^* e & -c^* e & 0 \end{bmatrix}$$

(I) *here* $e = \sqrt{k_1^2 + k_2^2} \neq 0.$

some results

Proof.

Let $[\eta] = [A]\,[V]$ and $[\eta^*] = [A^*]\,[V^*]$, hence

$$\begin{aligned}
[\eta]\,[\eta^*]^T &= [A]\,[V]\,([A^*]\,[V^*])^T \\
&= [A]\left([V]\,[V^*]^T\right)[A^*]^T \\
&= \frac{1}{\sqrt{k_1^2+k_2^2}}[A]\begin{bmatrix} 0 & -k_1 & k_2 \\ \sqrt{k_1^2+k_2^2} & 0 & 0 \\ 0 & k_2 & k_1 \end{bmatrix}[A^*]^T .
\end{aligned}$$

□

some results

Proof.

The matrix product

$$[\eta]\,[\eta^*]^T = \begin{bmatrix} 0 & \frac{-k_2}{\sqrt{k_1^2+k_2^2}} & \frac{-k_1}{\sqrt{k_1^2+k_2^2}} \\ 0 & \frac{-ak_1+bk_2}{\sqrt{k_1^2+k_2^2}} & \frac{ak_2+bk_1}{\sqrt{k_1^2+k_2^2}} \\ d & \frac{-ck_1}{\sqrt{k_1^2+k_2^2}} & \frac{ck_2}{\sqrt{k_1^2+k_2^2}} \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \\ 0 & -1 & 0 \end{bmatrix}^T$$

give us the result. □

some results

Theorem

In the Euclidean 3 − space, the position of the unit normal vector field $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of ruled surfaces $\varphi^{*1}, \varphi^{*2}, \varphi^{*3}, \varphi^{*4}$, recpectively, along the curve involute α^*, can be expressed by the following equations; $[\eta]\,[\eta^*]^T = \begin{bmatrix} \langle\eta_1,\eta_1^*\rangle & \langle\eta_1,\eta_2^*\rangle & \langle\eta_1,\eta_3^*\rangle & \langle\eta_1,\eta_4^*\rangle \\ \langle\eta_2,\eta_1^*\rangle & \langle\eta_2,\eta_2^*\rangle & \langle\eta_2,\eta_3^*\rangle & \langle\eta_2,\eta_4^*\rangle \\ \langle\eta_3,\eta_1^*\rangle & \langle\eta_3,\eta_2^*\rangle & \langle\eta_3,\eta_3^*\rangle & \langle\eta_3,\eta_4^*\rangle \\ \langle\eta_4,\eta_1^*\rangle & \langle\eta_4,\eta_2^*\rangle & \langle\eta_4,\eta_3^*\rangle & \langle\eta_4,\eta_4^*\rangle \end{bmatrix}$ (II).

Proof.

It is trivial with the product of matrices

$$[\eta]\,[\eta^*]^T = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \ldots \end{bmatrix} \begin{bmatrix} \eta_1^* & \eta_2^* & \eta_3^* & \eta_4^* \end{bmatrix}.$$

□

some results

In the Euclidean $3-space$, the position of two surface, basicly, can be examined by the position of their unit normal vector fields. According the equalities of the last matrice we have the simple results as in the following theorems. In this section using the equality of the matrices (I) and (II), we give sixteen interesting results according to the normal vector fields with the following theorems.

some results

Theorem

Tangent ruled surface and involutive tangent ruled surface of the evolute α have normal vector fields with the condition $\cos\phi = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$ *here the angel between* η_1 *and* η_1^* *is* ϕ*, (a nonzero function of the curvatures* k_1 *and* k_2 *of the evolute* α*).*

Proof.

Since $\langle \eta_1, \eta_1^* \rangle = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$, it is trivial. □

some results

Theorem

Tangent ruled surface and involutive normal ruled surface of the evolute α have perpendicular along the curve

$$\varphi^{*2}(s) = \alpha(s) + \left(\lambda - \frac{\lambda k_1^2}{k_1^2 + k_2^2}\right) V_1(s) + \frac{\lambda k_1 k_2}{k_1^2 + k_2^2} V_3(s).$$

Proof.

Since $\langle \eta_1, \eta_2^* \rangle = \frac{-k_1(1-v_2 k_1^*)}{e\sqrt{(v_2 k_2^*)^2 + (1-v_2 k_1^*)^2}}$,and for $\langle \eta_1, \eta_2^* \rangle = 0$ we have perpendicular normal vector fields under the condition $v_2 = \frac{\lambda k_1}{\sqrt{k_1^2 + k_2^2}}$ we have the proof. □

some results

Theorem

Tangent ruled surface and involutive binormal ruled surface of the evolute α have not perpendicular normal vector fields, except $\lambda = 0$.

Proof.

Since $\langle \eta_1, \eta_3^* \rangle = \frac{-k_2 d^*}{\sqrt{k_1^2+k_2^2}}$, and for $\langle \eta_1, \eta_3^* \rangle = 0$ we have $\lambda k_1 k_2 \left(k_1^2 + k_2^2\right) = 0$. □

some results

Theorem

Tangent ruled surface and involutive Darboux ruled surface of the evolute α have normal vector fields with the condition $\cos\phi = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$ *here the angel between η_1 and η_4^* is ϕ .*

Proof.

Since $\langle \eta_1, \eta_4^* \rangle = \frac{k_2}{\sqrt{k_1^2+k_2^2}}$,it is trivial. □

some results

Theorem

Normal ruled surface and involutive tangent ruled surface of the evolute α have the perpendicular normal vector fields along the curve $\varphi^2(s) = \alpha(s) + \frac{-k_1}{k_1^2+k_2^2} V_2(s)$.

Proof.

Since $\langle \eta_2, \eta_1^* \rangle = \frac{-ak_2-bk_1}{\sqrt{k_1^2+k_2^2}}$, and for $\langle \eta_2, \eta_1^* \rangle = 0$ we have perpendicular normal vector fields under the condition $u_2 = \frac{-k_1}{k_1^2+k_2^2}$. We have the proof. □

some results

Theorem

Normal ruled surface and involutive normal ruled surface of the evolute α have the perpencicular normal vector fields along the curve $\varphi^{*2}(s) = \alpha(s) + \left(\lambda - \frac{\lambda k_1^2}{k_1^2+k_2^2}\right) V_1(s) + \frac{\lambda k_1 k_2}{k_1^2+k_2^2} V_3(s)$.

Proof.

Since $\langle \eta_2, \eta_2^* \rangle = \frac{(ak_2+bk_1)b^*}{\sqrt{k_1^2+k_2^2}}$, and for $\langle \eta_2, \eta_2^* \rangle = 0$, we have the perpencicular normal vector fields under the condition $v_2 = \frac{(\sigma - s)k_1}{\sqrt{k_1^2+k_2^2}}$, $(\sigma - s)k_1 > 0$, $k_1 \neq 0$, this complete the proof. □

some results

Theorem

Normal ruled surface and involutive binormal ruled surface of the evolute α have not the perpencicular normal vector fields, except $(\sigma - s) = 0$

Proof.

Since $\langle \eta_2, \eta_3^* \rangle = \frac{(-ak_1 + bk_2)d^*}{\sqrt{k_1^2 + k_2^2}}$ *and for* $\langle \eta_2, \eta_3^* \rangle = 0$ *it is trivial.* □

some results

Theorem

Normal ruled surface and involutive Darboux ruled surface of the evolute α have not the perpencicular normal vector fields unless $k_2 \neq 0$.

Proof.

Since $\langle \eta_2, \eta_4^* \rangle = \frac{-(-ak_1+bk_2)}{e}$ *and for* $\langle \eta_2, \eta_4^* \rangle = 0$ *it is trivial.* □

some results

Theorem

Binormal ruled surface and involutive tangent ruled surface of the evolute α have the perpencicular normal vector fields only along the directrix of involute α^*.

Proof.

Since $\langle \eta_3, \eta_1^* \rangle = \frac{-k_2 c}{e}$, *and for* $\langle \eta_3, \eta_1^* \rangle = 0$, *under the condition* $u_3 = 0$, *it completes the proof.* □

some results

Theorem

Binormal ruled surface and involutive normal ruled surface of the evolute α have the perpencicular normal vector fields under the condition $v_2 = \frac{u_3 \lambda k_1 k_2^2 \sqrt{k_1^2 + k_2^2}}{k_2' k_1 - k_1' k_2 + u_3 k_2^2 \left(k_1^2 + k_2^2\right)}$.

Proof.

Since $\langle \eta_3, \eta_2^* \rangle = \frac{eda^* + k_2 cb^*}{e}$ and for $\langle \eta_3, \eta_2^* \rangle = 0$ it can be calculated easily. Where u_3 and v_2 are the parameters of binormal ruled surface and *involutive normal ruled surface* of the evolute α, respectively. □

some results

Theorem

Binormal ruled surface and involutive binormal ruled surface of the evolute α have the perpencicular normal vector fields under the condition $v_3 = u_3 \frac{\lambda k_1^2 \sqrt{k_1^2 + k_2^2}}{-k_2 \left(\frac{k_1}{k_2}\right)'}$.

Proof.

Since $\langle \eta_3, \eta_3^* \rangle = \frac{edc^* - k_1 cd^*}{\sqrt{k_1^2 + k_2^2}}$ and for $\langle \eta_3, \eta_3^* \rangle = 0$ it can be calculated easily. □

some results

Theorem

Binormal ruled surface and involutive Darboux ruled surface of the evolute α have not the perpencicular normal vector fields except $u_3 = 0$.

Proof.

Since $\langle \eta_3, \eta_4^* \rangle = \frac{k_1 c}{\sqrt{k_1^2 + k_2^2}}$ *and for* $\langle \eta_3, \eta_4^* \rangle = 0$ *it is trivial.* □

some results

Theorem

Darboux ruled surface and involutive tangent ruled surface of an evolute α have the perpendicular normal vector fields.

Proof.

Since $\langle \eta_4, \eta_1^* \rangle = 0$ it is trivial. □

some results

Theorem

Darboux ruled surface and involutive normal ruled surface of an evolute α have the perpendicular normal vector fields, under the condition $v_2 \frac{-k_2^2\left(\frac{k_1}{k_2}\right)'}{\lambda k_1\left(k_1^2+k_2^2\right)} = 0$

Proof.

Since $\langle \eta_4, \eta_2^* \rangle = -a^* e$ and for $\langle \eta_4, \eta_2^* \rangle = 0$,we get the proof. □

some results

Theorem

Darboux ruled surface and involutive binormal ruled surface of an evolute α have the perpendicular normal vector fields under the condition $v_3 \frac{-k_2^2\left(\frac{k_1}{k_2}\right)'}{\lambda k_1\left(k_1^2+k_2^2\right)} = 0.$

Proof.

Since $\langle \eta_4, \eta_3^* \rangle = -c^* e$ and for $\langle \eta_4, \eta_3^* \rangle = 0$ we get the proof. □

some results

Theorem

Darboux ruled surface and involutive Darboux ruled surface of an evolute α have the perpendicular normal vector fields.

Proof.

Since $\langle \eta_4, \eta_4^* \rangle = 0$.it is trivial. □

References

- Boyer C., *A History of Mathematics*, Wiley, New York 1968.
- do Carmo, M. P., Differential Geometry of Curves and Surfaces. Prentice-Hall, ISBN 0-13-212589-7, 1976.
- Eisenhart, Luther P., *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover, ISBN 0-486-43820-1 (2004).
- Graves L.K., *Codimension one isometric immersions between Lorentz spaces.* Trans. Amer. Math. Soc. **252** (1979) 367–392.
- Gray, A. *Modern Differential Geometry of Curves and Surfaces with Mathematica,* 2nd ed. Boca Raton, FL: CRC Press, p. 205, 1997.
- Hacisalihoğlu H.H., *Diferensiyel Geometri*, Cilt 1, İnönü Üniversitesi Yayinlari, Malatya 1994.

References

- Kiliçoğlu Ş., *n-Boyutlu Lorentz uzayinda B-scrollar.* Doktora tezi, Ankara Üniversitesi Fen Bilimleri Enstitüsü, Ankara 2006.
- Kılıcoglu Ş. ;On the b-scrolls with time-like directrix in 3-dimensional Minkowski Space. *Beykent University Journal of Science and Technology*, 2008; 2(2):206-.
- Kılıcoglu Ş. ;On the generalized B-scrolls with p th degree in n- dimensional Minkowski space and striction (central spaces). *Sakarya Üniversitesi Fen Edebiyat Dergisi,* 10(2) , 15-29 (ISSN: 1301-3769)., 2008; 10(2):15-29.
- Kılıçoğlu Ş, H. Hilmi Hacisalihoglu and Senyurt S., On the fundamental forms of the B-scroll with null directrix and Cartan frame in Minkowskian 3-space. *Applied Mathematical Sciences,* Vol. 9, 2015, no. 80, 3957 - 3965.

References

- Kılıçoğlu Ş. ;On the Involutive B-scrolls in the Euclidean Three-space E 3. XIII^{th} . *Geometry Integrability and Quantization*, Varna, Bulgaria: Sofia 2012,pp 205-214.
- Senyurt, S. and Kılıcoglu Ş, On the differential geometric elements of the *involute* $\tilde{D}$ scroll, Adv. Appl. Cliff ord Algebras 2015 Springer Basel, doi:10.1007/s00006-015-0535-z.
- L.J. Alias, Angel Ferrandez, Pascual Lucas and Miguel Angel Merono, On the Gauss map of B-scrolls, Tsukuba J. Math. 22 (1998), 371-377.
- Izumiya, S.; Takeuchi, N.: Special curves and Ruled surfaces . Beitr¨age zur Algebra und Geometrie Contributions to Algebra and Geometry, Volume 44 (2003), No. 1, 203-212.
- Lipschutz M.M., *Differential Geometry*, Schaum's Outlines

Thank You for Your Attention