$XVII^{th}$ International Conference of
Geometry, Integrability and Quantization
Sveti Konstantin i Elena 2016

On the existence of isoperimetric extremals of rotation and the fundamental equations of rotary diffeomorphisms

Hana CHUDÁ, Josef MIKEŠ

Tomas Bata University Zlín
Palacký University Olomouc

7. června 2016

The isoperimetric extremals of rotation

We suppose that $V_2 = (M, g)$ is a (pseudo-) Riemannian manifold belongs to the smoothness class C^r if its metric $g \in C^r$, i.e. its components $g_{ij}(x) \in C^r(U)$ in some local map (U, x) $U \subset M$.

Differentiability class r is equal to $0, 1, 2, \dots, \infty, \omega$, where $0, \infty$ and ω denote continuous, infinitely differentiable, and real analytic functions, respectively.

The isoperimetric extremals of rotation

Let $\ell\colon (s_0, s_1) \to M$ be a **parametric curve** with the equation $x = x(s)$, we construct $\lambda = \mathrm{d}x/\mathrm{d}s$ the tangent vector and s is the arc length.

The following formulas were developed by analogy with the Frenet formulas for a manifold V_2:

$$\nabla_s \lambda = k \cdot n \quad \text{and} \quad \nabla_s n = -\varepsilon\, \varepsilon_n\, k \cdot \lambda,$$

in these equations represents k the Frenet curvature; n is the unit vector field along ℓ, orthogonal to the tangent vector λ, ∇_s is an operator of covariant derivative along ℓ with the respect to the Levi-Civita connection ∇; ε, ε_n are constants ± 1.

The isoperimetric extremals of rotation

That is to say

$$\nabla_s \lambda^h = \frac{\mathrm{d}\lambda^h}{\mathrm{d}s} + \lambda^\alpha \Gamma^h_{\alpha\beta}\left(x(s)\right) \frac{\mathrm{d}x^\beta(s)}{\mathrm{d}s},$$
$$\nabla_s n^h = \frac{\mathrm{d}n^h}{\mathrm{d}s} + n^\alpha \Gamma^h_{\alpha\beta}\left(x(s)\right) \frac{\mathrm{d}x^\beta(s)}{\mathrm{d}s},$$

where $\Gamma^h_{\alpha\beta}$ are the Christoffel symbols of V_2, i. e. components of Levi-Civita connection ∇; λ^h and n^h are components of the vectors λ and n. We suppose that λ and n are non-isotropic vectors and

$$\langle \lambda, \lambda \rangle = g_{ij}\lambda^i\lambda^j = \varepsilon = \pm 1,$$
$$\langle n, n \rangle = g_{ij}n^i n^j = \varepsilon_n = \pm 1.$$

The isoperimetric extremals of rotation

Remind assertion for the **scalar product** of the vectors λ, ξ is defined by

$$\langle \lambda, \xi \rangle = g_{ij} \lambda^i \xi^j.$$

Denote

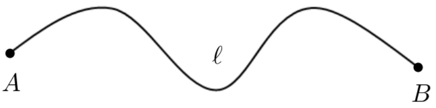
$$s[\ell] = \int_{s_0}^{s_1} \sqrt{|\lambda|}\, \mathrm{d}s \qquad \text{and} \qquad \theta[\ell] = \int_{s_0}^{s_1} k(s)\, \mathrm{d}s$$

functionals of **length** and **rotation** of the curve ℓ: $x = x(s)$. And $|\lambda| = |g_{\alpha\beta} \lambda^\alpha \lambda^\beta|$ is the length of a vector λ and k(s) is the Frenet curvature.

The isoperimetric extremals of rotation

Definition

A curve ℓ is called the **isoperimetric extremal of rotation** if ℓ is extremal of $\theta[\ell]$ and $s[\ell] = \text{const}$ with fixed ends.



$$s[\ell] = \int_{s_0}^{s_1} \sqrt{|\lambda|}\ \mathrm{d}s \qquad\qquad \theta[\ell] = \int_{s_0}^{s_1} k(s)\ \mathrm{d}s$$

$$s[\ell] = \text{const}$$

The isoperimetric extremals of rotation

G.S. Leiko [2] proved

Theorem

A curve ℓ is an isoperimetric extremal of rotation only and only if, its Frenet curvature k and Gaussian curvature K are proportional

$$k = c \cdot K,$$

where $c = \text{const}$.

The isoperimetric extremals of rotation

J. Mikeš, M. Sochor and E. Stepanova proved the following

Theorem

The equation of isoperimetric extremal of rotation can be written in the form

$$\nabla_s \lambda = c \cdot K \cdot F\lambda \tag{1}$$

where $c = \text{const}$.

It can be easily proved that vector $F\lambda$ is also a unit vector orthogonal to a unit vector λ and if $c = 0$ is satisfied then the curve is geodesic.

Structure F is the tensor (1, 1) which satisfies the following conditions (in the invariant form)

The isoperimetric extremals of rotation

$$F^2 = e \cdot \mathrm{Id}, \quad g(X, FX) = 0, \quad \nabla F = 0.$$

- for a Riemannian manifold $V_2 \in C^2$ is $e = -1$ and F is a *complex structure*.
- for (pseudo-) Riemannian manifold is $e = +1$ and F is *product structure*.

The tensor F is uniquely defined with using skew-symmetric and covariantly constant discriminant tensor ε.

$$F_i^h = \varepsilon_{ij} \cdot g^{jh} \qquad \text{and} \qquad \varepsilon_{ij} = \sqrt{|g_{11}g_{22} - g_{12}^2|} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The isoperimetric extremals of rotation

Analysis of the equation (1) convinces of the validity of the following theorem which generalizes and refines the results by G.S. Leiko.
J. Mikeš proved

Theorem

Let V_2 be a (non-flat) Riemannian manifold of the smoothness class C^3. Then there is precisely one isoperimetric extremal of rotation going through a point $x_0 \in V_2$ in a given non-isotropic direction $\lambda_0 \in TV_2$ and constant c.

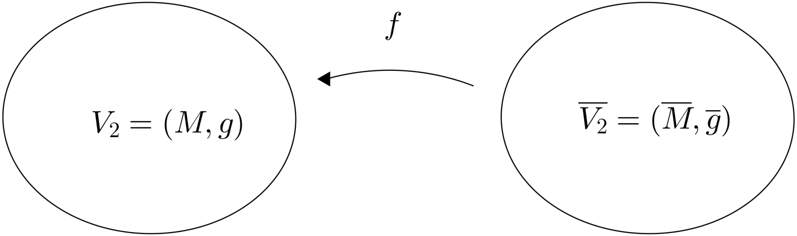
The fundamental equations of rotary diffeomorphisms

Assume to be given two - dimensional (pseudo-) Riemannian manifolds $V_2 = (M, g)$ and $\overline{V}_2 = (\overline{M}, \overline{g})$ with metrics g and $\overline{g}$, Levi-Civita connections ∇ and $\overline{\nabla}$, structures F and $\overline{F}$, respectively.

The fundamental equations of rotary diffeomorphisms

Definition

A diffeomorphism $f : \overline{V}_2 \to V_2$ is called **rotary** if any geodesic $\overline{\gamma}$ is mapped onto isoperimetric extremal of rotation of manifold V_2.



The fundamental equations of rotary diffeomorphisms

Assume a rotary diffeomorphism $f\colon \overline{V}_2 \to V_2$. Since f is a diffeomorphism, we can impose local coordinate system on M and $\overline{M}$, respectively, such that locally $f\colon \overline{V}_2 \to V_2$ maps points onto points with the same coordinates x, and $M \equiv \overline{M}$.

It holds

Theorem

Let $\overline{V}_2$ admits rotary mapping onto V_2. If V_2 and $\overline{V}_2$ belong to class C^2, then Gaussian curvature K on manifold V_2 is differentiable.

The fundamental equations of rotary diffeomorphisms

Let $\overline{\gamma}\colon x = x(\overline{s})$ be a geodesic on $\overline{V}_2$ for which the following equation is valid

$$\frac{d^2x^h}{d\overline{s}^{\,2}} + \overline{\Gamma}{}^h_{ij}(x(\overline{s}))\ \frac{dx^i}{d\overline{s}}\ \frac{dx^j}{d\overline{s}} = 0 \tag{2}$$

and let $\gamma\colon x = x(s)$ be an isoperimetric extremal of rotation on V_2 for which the following equation is valid

$$\frac{d\lambda^h}{ds} + \Gamma^h_{ij}(x(s))\ \lambda^i\lambda^j = c\cdot K(x(s))\cdot F^h_i(x(s))\cdot\lambda^i, \tag{3}$$

where Γ^h_{ij} and $\overline{\Gamma}{}^h_{ij}$ are components of ∇ and $\overline{\nabla}$, parameters s and $\overline{s}$ are arc lengthes on γ and $\overline{\gamma}$, $\lambda^h = dx^h(s)/ds$ and $\overline{\lambda}^h = dx^h(s)/d\overline{s}$.

The fundamental equations of rotary diffeomorphisms

Evidently $\bar{s} = \bar{s}(s)$. We modify equation (2)

$$\frac{d\lambda^h}{ds} + \overline{\Gamma}^h_{ij}(x(s))\ \lambda^i\lambda^j = \varrho(s)\cdot\lambda^h, \tag{4}$$

where $\varrho(s)$ is a certain function of parameter s.

We denote

$$P^h_{ij}(x) = \Gamma^{\,h}_{ij}(x) - \overline{\Gamma}^h_{ij}(x)$$

the **deformation tensor** of connections ∇ and $\overline{\nabla}$ defined by the rotary diffeomorphism.

The fundamental equations of rotary diffeomorphisms

By subtraction equations (3) and (4) we obtain

$$P^h_{ij}(x)\lambda^i\lambda^j = c \cdot K(x(s)) \cdot F^h_i(x(s)) \cdot \lambda^i - \varrho(s) \cdot \lambda^h \tag{5}$$

Because $\overline{V}_2 \in C^2$ from formulas (4) follows that $\varrho(s) \in C^1$. After differentiation formulas (5) we obtain that $K(x(s)) \in C^1$. And because these properties apply in any direction, then K is differentiable.

The fundamental equations of rotary diffeomorphisms

We proved the following

Theorem

If Gaussian curvature $K \notin C^1$ *then rotary diffeomorphism* $f : \overline{V}_2 \to V_2$ *does not exist.*

The fundamental equations of rotary diffeomorphisms

Assume that for the rotary diffeomorphism $f : \overline{V}_2 \to V_2$ formulas (5) hold. Contracting equations (5) with $g_{hi}\lambda^i$ we obtain

$$-e\,\varepsilon\,c\,K = F_i^h\,\lambda^i\,P^h_{\alpha\beta}\,\lambda^\alpha\lambda^\beta. \tag{6}$$

After subsequent differentiation of equations (6) along the curve, we obtain (7)

$$-e\,\varepsilon\,c\,K_{,\delta}\,\lambda^\delta = \varepsilon_{\gamma h}\,P^h_{\alpha\beta,\delta}\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta + c\,K\,\varepsilon_{\gamma h}\,P^h_{\alpha\beta}(2F^\beta_\delta\lambda^\alpha\lambda^\delta\lambda^\gamma + F^\gamma_\delta\lambda^\alpha\lambda^\beta\lambda^\delta).$$

The fundamental equations of rotary diffeomorphisms

The part of equation (7), which contains the components of the sixth degree, has the form

$$I = I_1 \cdot I_2, \tag{8}$$

where $I_1 = c \cdot K$; $I_2 = \varepsilon_{\gamma h}\, P^h_{\alpha\beta}(2F^\beta_\delta \lambda^\alpha \lambda^\delta \lambda^\gamma + F^\gamma_\delta \lambda^\alpha \lambda^\beta \lambda^\delta)$.

At the point x_0 we can metric tensor diagonalize, such that $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$, where $\varepsilon = \pm 1$ (by the metric signature).

Thus from condition $|\lambda| = 1$ follows that $(\lambda^2)^2 = \varepsilon e - e(\lambda^1)^2$.

The fundamental equations of rotary diffeomorphisms

With detailed analysis the highest degrees of $(\lambda^1)^6$ in the equation (8), we get

$$A = 3e \cdot C \quad B = 3e \cdot D, \tag{9}$$

where

$$A = 3e(-2P^2_{12} - P^1_{11} + eP^2_{22}), \quad B = 3e(eP^2_{11} - P^2_{22} - 2P^1_{12}),$$

$$C = (-2P^2_{12} - P^1_{11} + eP^2_{22}), \quad D = (eP^2_{11} - P^2_{22} - 2P^1_{12}).$$

The fundamental equations of rotary diffeomorphisms

From this it follows $A^2 + B^2 = 0$, and evidently $A = B = 0$, and hence we have in this coordinate system

$$-2P_{12}^2 - P_{11}^1 + eP_{22}^2 = 0 \quad \text{and} \quad eP_{11}^2 - P_{22}^2 - 2P_{12}^1 = 0.$$

We denote $\psi_1 = P_{12}^2$, $\psi_2 = P_{12}^1$, $\theta^1 = P_{22}^2$ and $\theta^2 = P_{11}^2$. We can rewrite the above mentioned formula (5) equivalently to the following tensor equation

$$P_{ij}^h = \delta_i^h \psi_j + \delta_j^h \psi_i + \theta^h g_{ij}, \tag{10}$$

where ψ_i and θ^h are covectors and vector fields.

The fundamental equations of rotary diffeomorphisms

On the other hand, we use the same analysis, as in the previous part for this equation

$$\theta^h = c\,\varepsilon\,K\;F_i^h\lambda^i - \varepsilon\,(\overline{\varrho} - 2\psi_\alpha\lambda^\alpha)\;\lambda^h \tag{11}$$

Further, obtain the following

$$\xi^h \equiv \theta^h\;F_\alpha^h = e\,\varepsilon\,c\,K\;\lambda^h - \varepsilon\,(\overline{\varrho} - 2\psi_\alpha\lambda^\alpha)\;\lambda^h F_\alpha^h$$

and after differentiation along curve ℓ and with analysis after the highest degrees we get

$$\nabla_\alpha\theta^h\lambda^\alpha - \theta^h\theta_\alpha\lambda^\alpha - \theta^h K_\alpha\lambda^\alpha/K = \lambda^h\,(\nabla_\beta\theta_\alpha\lambda^\alpha\lambda^\beta - \theta_\alpha\theta_\beta\lambda^\alpha\lambda^\beta - \theta_\alpha\lambda^\alpha\;K_\beta\lambda^\beta/$$

and by similar way we obtain the following formulas

$$\nabla_j\theta_i = \theta_i(\theta_j + K_j/K) + \nu\,g_{ij}, \tag{12}$$

where ν is a function on V_2.

The fundamental equations of rotary diffeomorphisms

Theorem

The equations

$$P_{ij}^h = \delta_i^h \psi_j + \delta_j^h \psi_i + \theta^h g_{ij}, \qquad (10)$$

$$\nabla_j \theta_i = \theta_i(\theta_j + K_j/K) + \nu\, g_{ij}, \qquad (12)$$

where ψ_i, θ^h are covector and vector fields, ν is a function on V_2, are necessary and sufficient conditions of rotary diffeomorphism V_2 onto $\bar{V}_2$.

This proof is straightforward than the one proposed by G.S. Leiko [3]. We note that the above considerations are possible when $V_2 \in C^2$ and $\overline{V}_2 \in C^2$.

The fundamental equations of rotary diffeomorphisms

The vector field θ_i is torse-forming. Under further conditions on differentiability of the metrics it has been proved in [3] that θ_i is concircular.

Theorem

Let $f : \overline{V}_2 \to V_2$ be a rotary diffeomorphism, metrics of (pseudo-) Riemannian manifolds V_2 and $\overline{V}_2$ have differentiability class C^2 on own coordinate system; then manifolds V_2 and $\overline{V}_2$ are equidistant and isometric of revolution surfaces and metrics are positive definite.

References

- A. Gray, *Modern differential geometry of curves and surfaces with Mathematica.* 2nd ed. Boca Raton, FL: CRC Press, 1997.
- S. G. Leiko, *Conservation laws for spin trajectories generated by isoperimetric extremals of rotation.* Gravitation and Theory of Relativity **26** (1988), 117–124.
- S. G. Leiko, *Rotary diffeomorphisms on Euclidean spaces.* Mat. Zametki **47:3** (1990), 52–57.
- J. Mikeš, M. Sochor, E. Stepanova *On the existence of isoperimetric extremals of rotation and the fundamental equations of rotary diffeomorphisms.* Filomat, (2015).
- S. M. Minčić, Lj. S. Velimirović, M. S. Stanković., *Integrability conditions of derivational equations of a submanifold of a generalized Riemannian space.* Applied Mathematics and Computation **226:1** (2014), 3–9.
- I. V. Potapenko, *Leiko network on the surfaces in the Euclidean space E_3.* Ukrainian Math. J. 67 (2015), no. 6, 928–937.

Thank You for Your time.