Biharmonic pmc surfaces in complex space forms

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Harmonic and biharmonic maps

Let $\varphi : (M,g) \to (N,h)$ be a smooth map.

Energy functional

$$E(\varphi) = E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned} \tau(\varphi) = \tau_1(\varphi) &= \operatorname{trace}_g \nabla d\varphi \\ &= 0 \end{aligned}$$

Critical points of E:
harmonic maps

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Critical points of E:
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Bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

Euler-Lagrange equation

$$\begin{aligned}\tau_2(\varphi) &= \Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi \\ &= 0\end{aligned}$$

Critical points of E_2:
biharmonic maps

The biharmonic equation (Jiang, 1986)

$$\tau_2(\varphi) = \Delta^{\varphi}\tau(\varphi) - \operatorname{trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0$$

where

$$\Delta^{\varphi} = \operatorname{trace}_g\left(\nabla^{\varphi}\nabla^{\varphi} - \nabla^{\varphi}_{\nabla}\right)$$

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- is a fourth-order non-linear elliptic equation
- any harmonic map is biharmonic
- a non-harmonic biharmonic map is called proper biharmonic
- the biharmonic submanifolds M of a given space N are the submanifolds such that the inclusion map $i : M \to N$ is biharmonic (the inclusion map $i : M \to N$ is harmonic if and only if M is minimal)

The biharmonic equation

Theorem (Balmuş-Montaldo-Oniciuc, 2012)

A submanifold Σ^m in a Riemannian manifold N, with second fundamental form σ, mean curvature vector field H, and shape operator A, is biharmonic if and only if

$$\begin{cases} -\Delta^{\perp}H + \operatorname{trace}\sigma(\cdot, A_H\cdot) + \operatorname{trace}(R^N(\cdot,H)\cdot)^{\perp} = 0 \\ \frac{m}{2}\operatorname{grad}|H|^2 + 2\operatorname{trace}A_{\nabla^{\perp}_{\cdot}H}(\cdot) + 2\operatorname{trace}(R^N(\cdot,H)\cdot)^{\top} = 0, \end{cases}$$

where $\Delta^{\perp}$ is the Laplacian in the normal bundle.

Biharmonic submanifolds in Euclidean spaces

$$R^N = 0 \Rightarrow \tau_2(\varphi) = \Delta^\varphi \tau(\varphi)$$

Definition (Chen)

A submanifold $i : M \to \mathbb{R}^n$ is biharmonic if it has harmonic mean curvature vector field, i.e.,

$$\Delta^i H = 0 \Leftrightarrow \Delta^i \tau(i) = 0.$$

Non existence of proper biharmonic submanifolds

For any of the following classes of submanifolds the biharmonicity is equivalent to minimality:

- submanifolds of $N^3(c)$, $c \leq 0$ (Chen/Caddeo - Montaldo - Oniciuc)
- curves of $N^n(c)$, $c \leq 0$ (Dimitric/Caddeo - Montaldo - Oniciuc)
- submanifolds of finite type in $\mathbb{R}^n$ (Dimitric)
- hypersurfaces of $\mathbb{R}^n$ with at most two principal curvatures (Dimitric)
- pseudo-umbilical submanifolds of $N^n(c)$, $c \leq 0$, $n \neq 4$ (Dimitric/Caddeo - Montaldo - Oniciuc)
- hypersurfaces of $\mathbb{R}^4$ (Hasanis - Vlachos)
- spherical submanifolds of $\mathbb{R}^n$ (Chen)

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Chen's conjecture (still open)

Any biharmonic submanifold of the Euclidean space is minimal.

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Chen's conjecture (still open)

Any biharmonic submanifold of the Euclidean space is minimal.

Generalized Chen's Conjecture (still open)

Biharmonic submanifolds of $N^n(c)$, $n > 3$, $c \leq 0$, are minimal.

Main examples of biharmonic submanifolds in $\mathbb{S}^n$ (Jiang, 1986/ Caddeo - Montaldo - Oniciuc, 2002)

The composition property

$$\mathbb{S}^{n-1}(a) \xrightarrow{\text{biharmonic}} \mathbb{S}^n \quad \Longleftrightarrow \quad a = \frac{1}{\sqrt{2}}$$

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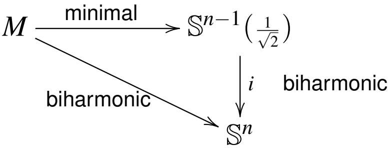

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Properties

- M has parallel mean curvature vector field and $|H| = 1$
- M is pseudo-umbilical in $\mathbb{S}^n$, i.e., $A_H = |H|^2 \operatorname{Id}$

Main examples of biharmonic submanifolds in $\mathbb{S}^n$

The product composition property

$$\mathbb{S}^{n_1}(a) \times \mathbb{S}^{n_2}(b) \xrightarrow{\text{biharmonic}} \mathbb{S}^n \iff a = b = \frac{1}{\sqrt{2}} \quad \text{and} \quad n_1 \neq n_2$$

$n_1 + n_2 = n - 1$, $a^2 + b^2 = 1$

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$n_1 + n_2 = n - 1$, $m_1 \neq m_2$

Properties

- $M_1 \times M_2$ has parallel mean curvature vector field and $|H| \in (0,1)$
- $M_1 \times M_2$ is not pseudo-umbilical in $\mathbb{S}^n$

Complex space forms

Definition

A complex space form is a $2n$-dimensional Kähler manifold $N^n(\rho)$ of constant holomorphic sectional curvature ρ.

A complex space form $N^n(\rho)$ is either:

- the complex projective space $\mathbb{C}P^n(\rho)$, if $\rho > 0$
- the complex Euclidean space $\mathbb{C}^n$, if $\rho = 0$
- the complex hyperbolic space $\mathbb{C}H^n(\rho)$, if $\rho < 0$

The curvature tensor

$$\begin{aligned} R^N(U,V)W &= \tfrac{\rho}{4}\{\langle V,W\rangle U - \langle U,W\rangle V + \langle JV,W\rangle JU - \langle JU,W\rangle JV \\ &\quad +2\langle JV,U\rangle JW\} \end{aligned}$$

Biharmonic submanifolds of $\mathbb{C}P^n$

Let $i : \Sigma^m \to N^n(\rho)$ be a submanifold of real dimension m.

- (Gauss) $\nabla^N_X Y = \nabla_X Y + \sigma(X,Y)$
- (Weingarten) $\nabla^N_X V = -A_V X + \nabla^\perp_X V$

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$N^n(\rho) = \mathbb{C}P^n(4)$

The biharmonic equation is

$$\tau_2(i) = m\{\Delta H + mH - 3J\,(JH)^\top\} = 0$$

Biharmonic submanifolds of $\mathbb{C}P^n$

Proposition

If JH is tangent to Σ^m, then Σ^m is biharmonic iff

$$\begin{cases} -\Delta^{\perp} H + \operatorname{trace}\sigma(\cdot, A_H(\cdot)) - (m+3)H = 0 \\ 4\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) + m\operatorname{grad}(|H|^2) = 0. \end{cases}$$

Theorem (F. - Loubeau - Montaldo - Oniciuc, 2010)

If JH is tangent to Σ^m and $|H| = \text{constant} \neq 0$, then

1. *If Σ^m is proper-biharmonic, then $|H|^2 \in (0, \frac{m+3}{m}]$.*
2. *If $|H|^2 = \frac{m+3}{m}$, then Σ^m is proper-biharmonic iff it is pseudo umbilical, i.e., $A_H = |H|^2 \operatorname{Id}$, and $\nabla^{\perp} H = 0$.*

Remark

The upper bound of $|H|^2$ is reached for curves.

Biharmonic submanifolds of $\mathbb{C}P^n$

Proposition

If JH is normal to Σ^m, then Σ^m is biharmonic if and only if

$$\begin{cases} -\Delta^{\perp} H + \operatorname{trace} \sigma(\cdot, A_H(\cdot)) - mH = 0 \\ 4 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + m \operatorname{grad}(|H|^2) = 0. \end{cases}$$

Moreover, if Σ^m has parallel mean curvature, i.e., $\nabla^{\perp} H = 0$, then it is biharmonic iff

$$\operatorname{trace} \sigma(\cdot, A_H(\cdot)) = mH.$$

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If JH is normal to Σ^m and $|H| = \text{constant} \neq 0$, then

1. *If Σ^m is proper-biharmonic, then $|H|^2 \in (0,1]$.*
2. *If $|H|^2 = 1$, then Σ^m is proper-biharmonic iff it is pseudo-umbilical and $\nabla^{\perp} H = 0$.*

Remark

The upper bound is reached for curves.

The Hopf fibration and the biharmonic equation

- let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ be the natural projection.
- the restriction to the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ is the Hopf fibration

$$\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$$

- $\bar{\Sigma} = \pi^{-1}(\Sigma^m)$ is the Hopf tube over $\Sigma^m \subset \mathbb{C}P^n(4)$

$$\begin{array}{ccc} \bar{\Sigma} & \xrightarrow{\ \bar{i}\ } & \mathbb{S}^{2n+1} \\ \downarrow & & \downarrow \pi \\ \Sigma & \xrightarrow{\ i\ } & \mathbb{C}P^n \end{array}$$

The Hopf fibration and the biharmonic equation

Theorem (F.- Loubeau - Montaldo - Oniciuc, 2010)

Let $i:\Sigma^m \to \mathbb{C}P^n$ be an m-dimensional submanifold and $\bar{i}:\bar{\Sigma}=\pi^{-1}(\Sigma)\to\mathbb{S}^{2n+1}$ the corresponding Hopf-tube. Then we have

$$(\tau_2(i))^H = \tau_2(\bar{i}) - 4\hat{J}(\hat{J}\tau(\bar{i}))^\top + 2\,\mathrm{div}((\hat{J}\tau(\bar{i}))^\top)\xi$$

where $\bar{X}=X^H$ is the horizontal lift with respect to the Hopf map, ξ is the Hopf vector field on $\mathbb{S}^{2n+1}$ tangent to the fibres of the Hopf fibration, i.e., $\xi(p)=-\hat{J}p$ at any $p\in\mathbb{S}^{2n+1}$, and $\hat{J}$ is the complex structure of $\mathbb{R}^{2n+2}$.

Remark

- *If $\hat{J}\tau(\bar{i})$ is normal to $\bar{\Sigma}$, then $\tau_2(i)=0$ iff $\tau_2(\bar{i})=0$.*
- *If $\hat{J}\tau(\bar{i})$ is tangent to $\bar{\Sigma}$, then $\tau_2(i)=0$ and $\mathrm{div}_\Sigma((J\tau(i))^\top)=0$ iff $\tau_2(\bar{i})+4\tau(\bar{i})=0$.*

The Hopf fibration and the biharmonic equation

Theorem (Reckziegel, 1985)

A totally real isometric immersion $i : \Sigma^m \to \mathbb{C}P^n(\rho)$ can be lifted locally (or globally, if Σ^m is simply connected) to a horizontal immersion $\widetilde{i} : \widetilde{\Sigma}^m \to \mathbb{S}^{2n+1}(\rho/4)$. Conversely, if $\widetilde{i} : \widetilde{\Sigma}^m \to \mathbb{S}^{2n+1}(\rho/4)$ is a horizontal isometric immersion, then $\pi(\widetilde{i}) : \Sigma^m \to \mathbb{C}P^n(\rho)$ is a totally real isometric immersion. Moreover, we have $\pi_* \widetilde{\sigma} = \sigma$, where $\widetilde{\sigma}$ and σ are the second fundamental forms of the immersions $\widetilde{i}$ and i, respectively.

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Proposition (F.-Loubeau-Montaldo-Oniciuc, 2010)

Let $\widetilde{i}:\widetilde{\Sigma}^m \to \mathbb{S}^{2n+1}(\rho/4)$ be a horizontal isometric immersion and consider the totally real isometric immersion $i = \pi(\widetilde{i}):\Sigma^m \to \mathbb{C}P^n(\rho)$. Then

$$(\tau_2(i))^H = \tau_2(\widetilde{i}) - 4\hat{J}(\hat{J}\tau(\widetilde{i}))^\top + 2\,\mathrm{div}_{\widetilde{\Sigma}^m}((\hat{J}\tau(\widetilde{i}))^\top)\xi.$$

Curves in $\mathbb{C}P^n$

Definition

A curve $\gamma : I \subset \mathbb{R} \to \mathbb{C}P^n(\rho)$ parametrized by arc-length is called a Frenet curve of osculating order r, $1 \leq r \leq 2n$, if there exist r orthonormal vector fields $\{E_1 = \gamma', \ldots, E_r\}$ along γ such that

$$\begin{aligned} \nabla^{\mathbb{C}P^n}_{E_1} E_1 &= \kappa_1 E_2 \\ \nabla^{\mathbb{C}P^n}_{E_1} E_i &= -\kappa_{i-1} E_{i-1} + \kappa_i E_{i+1} \\ &\ldots \\ \nabla^{\mathbb{C}P^n}_{E_1} E_r &= -\kappa_{r-1} E_{r-1} \end{aligned}$$

for all $i \in \{2, \ldots, r-1\}$, where $\{\kappa_1, \kappa_2, \ldots, \kappa_{r-1}\}$ are positive functions on I called the curvatures of γ.

Curves in $\mathbb{C}P^n$

- a Frenet curve of osculating order r is called a helix of order r if $\kappa_i = \text{constant} > 0$ for $1 \leq i \leq r-1$. A helix of order 2 is called a circle, and a helix of order 3 is called a helix
- the complex torsions τ_{ij} of the curve γ are given by

$$\tau_{ij} = \langle E_i, JE_j \rangle$$

  where $1 \leq i < j \leq r$
- a helix of order r is called a holomorphic helix of order r if all its complex torsions are constant

The existence of holomorphic helices

Theorem (Maeda-Adachi, 1997)

For given positive constants κ_1, κ_2, and κ_3, there exist four equivalence classes of holomorphic helices of order 4 in $\mathbb{C}P^2(\rho)$ with curvatures κ_1, κ_2, and κ_3 with respect to holomorphic isometries of $\mathbb{C}P^2(\rho)$.

Theorem (Maeda-Adachi, 1997)

For any positive number κ and for any number τ, such that $|\tau| < 1$, there exits a holomorphic circle with curvature κ and complex torsion τ in any complex space form.

Biharmonic pmc surfaces in $\mathbb{C}P^n(\rho)$

Definition

If the mean curvature vector field H of a surface Σ^2 immersed in a complex space form is parallel in the normal bundle, i.e., $\nabla^\perp H = 0$, then Σ^2 is called a pmc surface.

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Theorem (F., 2012)

The $(2,0)$-part $Q^{(2,0)}$ of the quadratic form Q defined on a pmc surface $\Sigma^2 \subset N^n(\rho)$ by

$$Q(X,Y) = 8|H|^2\langle A_H X, Y\rangle + 3\rho\langle X,T\rangle\langle Y,T\rangle,$$

where $T = (JH)^\top$ is the tangent part of JH, is holomorphic.

Biharmonic pmc surfaces in $\mathbb{C}P^n(\rho)$

Theorem (F. - Pinheiro, 2015)

Let Σ^2 be a complete non-minimal pmc surface with non-negative Gaussian curvature K isometrically immersed in a complex space form $N^n(\rho)$, $\rho \neq 0$. Then one of the following holds:

1. *the surface is flat;*
2. *there exists a point $p \in \Sigma^2$ such that $K(p) > 0$ and $Q^{(2,0)}$ vanishes on Σ^2.*

Biharmonic pmc surfaces in $\mathbb{C}P^n(\rho)$

Sketch of the proof

- Consider a symmetric traceless operator S on Σ^2, given by

$$S = 8|H|^2 A_H + 3\rho \langle T, \cdot \rangle T - \left(\frac{3\rho}{2} |T|^2 + 8|H|^4 \right) \mathrm{Id}$$

- $\langle SX, Y \rangle = Q(X, Y) - \frac{\operatorname{trace} Q}{2} \langle X, Y \rangle$

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- $\langle SX, Y\rangle = Q(X,Y) - \frac{\operatorname{trace} Q}{2}\langle X, Y\rangle$
- [Cheng-Yau, 1977] $\Rightarrow \frac{1}{2}\Delta|S|^2 = 2K|S|^2 + |\nabla S|^2 \geq 0$
  where K is the Gaussian curvature of the surface

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- [Cheng-Yau, 1977] $\Rightarrow \frac{1}{2}\Delta|S|^2 = 2K|S|^2 + |\nabla S|^2 \geq 0$ where K is the Gaussian curvature of the surface
- $K \geq 0$ ($\Rightarrow \Sigma^2 =$ parabolic) $\Rightarrow |S|^2 =$ bounded $\Rightarrow$ Q.E.D.

Biharmonic pmc surfaces in $\mathbb{C}P^n(\rho)$

Proposition

Let Σ^2 be a pmc surface in a complex space form $(N(\rho), J, \langle , \rangle)$. Then Σ^2 is biharmonic iff

$$\operatorname{trace} \sigma(\cdot, A_H \cdot) = \frac{\rho}{4}\{2H - 3(JT)^{\perp}\} \quad \textit{and} \quad (JT)^{\top} = 0$$

where T is the tangent part of JH.

Remark

Proper-biharmonic pmc surfaces exist only in $\mathbb{C}P^n(\rho)$, since

$$0 < |A_H|^2 = \frac{\rho}{4}\{2|H|^2 + 3|T|^2\}$$

Biharmonic pmc surfaces in $\mathbb{C}P^n(\rho)$

Proposition (F. - Pinheiro, 2015)

If Σ^2 is a proper-biharmonic pmc surface in $\mathbb{C}P^n(\rho)$ then $T = (JH)^\top$ has constant length. Moreover, if $|T| = \text{constant} \neq 0$, then $\nabla T = 0$.

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Proposition (F. - Pinheiro, 2015)

If Σ^2 is a complete proper-biharmonic pmc surface in $\mathbb{C}P^n(\rho)$ with $K \geq 0$ and $T = 0$, then $n \geq 3$ and Σ^2 is pseudo-umbilical and totally real. Moreover, the mean curvature of Σ^2 is $|H| = \sqrt{\rho}/2$.

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Proposition (F. - Pinheiro, 2015)

If Σ^2 is a proper-biharmonic pmc surface in $\mathbb{C}P^n(\rho)$ then $T = (JH)^\top$ has constant length. Moreover, if $|T| = \text{constant} \neq 0$, then $\nabla T = 0$.

Proposition (F. - Pinheiro, 2015)

If Σ^2 is a complete proper-biharmonic pmc surface in $\mathbb{C}P^n(\rho)$ with $K \geq 0$ and $T = 0$, then $n \geq 3$ and Σ^2 is pseudo-umbilical and totally real. Moreover, the mean curvature of Σ^2 is $|H| = \sqrt{\rho}/2$.

Proposition (F. - Pinheiro, 2015)

If Σ^2 is a complete proper-biharmonic pmc surface in $\mathbb{C}P^n(\rho)$ with $K \geq 0$ and $T \neq 0$, then the surface is flat and $\nabla A_H = 0$.

Biharmonic pmc surfaces in $\mathbb{C}P^n(\rho)$

Theorem (Balmuş - Montaldo - Oniciuc, 2008)

Let Σ^m be a proper-biharmonic cmc submanifold in $\mathbb{S}^n(\rho/4)$ with mean curvature vector field H. Then $|H| \in (0, \sqrt{\rho}/2]$ and, moreover, $|H| = \sqrt{\rho}/2$ if and only if Σ^m is minimal in a small hypersphere $\mathbb{S}^{n-1}(\rho/2) \subset \mathbb{S}^n(\rho/4)$.

The classification theorem

Theorem (F. - Pinheiro, 2015)

Let Σ^2 be a complete proper-biharmonic pmc surface with non-negative Gaussian curvature in $\mathbb{C}P^n(\rho)$. Then Σ^2 is totally real and either

1. *Σ^2 is pseudo-umbilical and its mean curvature is equal to $\sqrt{\rho}/2$. Moreover, $\Sigma^2 = \pi(\widetilde{\Sigma}^2) \subset \mathbb{C}P^n(\rho)$, $n \geq 3$, where $\pi : \mathbb{S}^{2n+1}(\rho/4) \to \mathbb{C}P^n(\rho)$ is the Hopf fibration and the horizontal lift $\widetilde{\Sigma}^2$ of Σ^2 is a complete minimal surface in a small hypersphere $\mathbb{S}^{2n}(\rho/2) \subset \mathbb{S}^{2n+1}(\rho/4)$; or*
2. *Σ^2 lies in $\mathbb{C}P^2(\rho)$ as a complete Lagrangian proper-biharmonic pmc surface. Moreover, if $\rho = 4$, then* $$\Sigma^2 = \pi\left(\mathbb{S}^1\left(\sqrt{\frac{9\pm\sqrt{41}}{20}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{11\mp\sqrt{41}}{40}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{11\mp\sqrt{41}}{40}}\right)\right) \subset \mathbb{C}P^2(4)$$ *; or*
3. *Σ^2 lies in $\mathbb{C}P^3(\rho)$ and $\Sigma^2 = \gamma_1 \times \gamma_2 \subset \mathbb{C}P^3(\rho)$, where $\gamma_1 : \mathbb{R} \to \mathbb{C}P^2(\rho) \subset \mathbb{C}P^3(\rho)$ is a holomorphic helix of order 4 with curvatures $\kappa_1 = \sqrt{\frac{7\rho}{6}}$, $\kappa_2 = \frac{1}{2}\sqrt{\frac{5\rho}{42}}$, $\kappa_3 = \frac{3}{2}\sqrt{\frac{\rho}{42}}$ and complex torsions $\tau_{12} = -\tau_{34} = \frac{11\sqrt{14}}{42}$, $\tau_{23} = -\tau_{14} = \frac{\sqrt{70}}{42}$, $\tau_{13} = \tau_2 = 0$, and $\gamma_2 : \mathbb{R} \to \mathbb{C}P^3(\rho)$ is a circle with curvature $\kappa = \sqrt{\rho/2}$ and complex torsion $\tau_{12} = 0$. Moreover, γ_1 and γ_2 always exist and are unique up to holomorphic isometries.*

The classification theorem (the proof)

Case I: $T = (JH)^\top = 0$

- let $\pi : \mathbb{S}^{2n+1}(\rho/4) \to \mathbb{C}P^n(\rho)$ be the Hopf fibration and $\widetilde{\Sigma}^2$ the horizontal lift of Σ^2
- $\Sigma^2 =$ proper-biharmonic $\Rightarrow \Sigma^2 =$ pseudo-umbilical and totally real with $|H| = \sqrt{\rho}/2$
- [Reckziegel, 1985] $\Rightarrow \widetilde{\Sigma}^2 \subset \mathbb{S}^{2n+1}(\rho/4)$ is pseudo-umbilical and pmc
- [F. - L. - M. - O., 2010] and $T = 0 \Rightarrow \Sigma^2 =$ proper-biharmonic iff $\widetilde{\Sigma}^2 =$ proper-biharmonic
- [Balmuş - Montaldo - Oniciuc, 2008] $\Rightarrow \widetilde{\Sigma}^2 =$ minimal in a small hypersphere $\mathbb{S}^{2n}(\rho/2) \subset \mathbb{S}^{2n+1}(\rho/4)$

The classification theorem (the proof)

Case II: $T = (JH)^\top \neq 0$

- $\Sigma^2 =$ proper-biharmonic $\Rightarrow \Sigma^2 =$ totally real and flat with $\nabla A_H = 0$
- $U =$ normal, $U \perp H$, $U \perp J(T\Sigma^2)$, Ricci eq. $\Rightarrow [A_H, A_U] = 0$
- $\Sigma^2 \neq$ pseudo-umbilical ($|T| = \text{constant} \neq 0$) $\Rightarrow A_U = 0$
- consider the global orthonormal frame field $\{E_1 = T/|T|, E_2\}$ on Σ^2
  [F. - P., 2015] $\Rightarrow \nabla E_1 = 0$ and $\nabla E_2 = 0$
- if $JH =$ tangent ($|T| = |H|$), then $L = \text{span}\{JE_1, JE_2\} \subset N\Sigma^2$ is parallel ($\nabla^\perp L \subset L$), $J(T\Sigma^2 \oplus L) = T\Sigma^2 \oplus L$
  [Eschenburg - Tribuzy, 1993] $\Rightarrow \Sigma^2$ is a complete Lagrangian proper-biharmonic pmc surface in $\mathbb{C}P^2(\rho)$
- $\rho = 4$: [Sasahara, 2007] $\Rightarrow$ (2)

The classification theorem (the proof)

- if $JH \neq$ tangent ($|T| < |H|$), then
  $L = \operatorname{span}\{E_3 = JE_1, E_4 = JE_2, E_5 = \frac{1}{|N|}JN, E_6 = \frac{1}{|N|}N\} \subset N\Sigma^2$ is parallel,
  $J(T\Sigma^2 \oplus L) = T\Sigma^2 \oplus L$ (where $N = (JH)^\perp$)
  [Eschenburg - Tribuzy, 1993] $\Rightarrow \Sigma^2$ lies in $\mathbb{C}P^3(\rho)$
- $\Sigma^2 =$ totally real, Ricci eq., $\operatorname{trace}(A_H A_U) = (\rho/4)\{2\langle H, U\rangle - 3\langle JT, U\rangle\}$, $K = 0$
  $\Rightarrow |H| = \frac{\rho}{3}$ and

$$A_3 = \frac{1}{2}\sqrt{\frac{\rho}{3}}\begin{pmatrix} -\frac{11}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \frac{1}{2}\sqrt{\frac{\rho}{3}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_5 = -\frac{1}{2}\sqrt{\frac{5\rho}{3}}\begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_6 = 0 \tag{1}$$

- $\nabla E_1 = \nabla E_2 = 0$, de Rham Decomposition Theorem $\Rightarrow \Sigma^2 = \gamma_1 \times \gamma_2$
- (1) and [Maeda - Adachi, 1997] $\Rightarrow$ (3)

References

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Thank you