The Heisenberg group
in mathematics and physics

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Origins
of the Heisenberg(–Weyl) group

Roger Howe said:

> *An investigator might be able to get what he wanted out of a situation while overlooking the extra structure imposed by the Heisenberg group, structure which might enable him to get much more.*

The basic operators of differentiation $\frac{d}{dx}$ and multiplication by x in satisfy to the same Heisenberg commutation relations $[Q, P] = I$ as observables of momentum and coordinate in quantum mechanics.
We shall start from the general properties the Heisenberg group and its representations. Many important applications will follow.

The Symplectic Form

Let $n \geqslant 1$ be an integer. For complex vectors z, $w \in \mathbb{C}^n$, we define:

$$z\bar{w} = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n, \tag{1}$$

where $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n)$.
The following notion is central for Hamiltonian mechanics.

Definition

The *symplectic form* ω on $\mathbb{R}^{2n}$ is a function of two vectors such that:

$$\omega(x, y; x', y') = xy' - x'y, \qquad \text{where } (x, y), (x', y') \in \mathbb{R}^{2n}, \tag{2}$$

Check the following properties:

1. ω is anti-symmetric $\omega(x, y; x', y') = -\omega(x', y'; x, y)$.
2. ω is bilinear:

$$\begin{aligned} \omega(x, y; \alpha x', \alpha y') &= \alpha\omega(x, y; x', y'), \\ \omega(x, y; x' + x'', y' + y'') &= \omega(x, y; x', y') + \omega(x, y; x'', y''). \end{aligned}$$

Exercise

1. Let $z = x + iy$ and $w = x' + iy'$ then ω can be expressed through the complex inner product (1) as $\omega(x, y; x', y') = -\Im(z\bar{w})$.
2. The symplectic form on $\mathbb{R}^2$ is equal to $\det \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}$. Consequently it vanishes if and only if (x, y) and (x', y') are collinear.
3. Let $A \in \mathrm{SL}_2(\mathbb{R})$ be a real 2×2 matrix with the unit determinant. Define:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = A \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{3}$$

   Then, $\omega(\tilde{x}, \tilde{y}; \tilde{x}', \tilde{y}') = \omega(x, y; x', y')$. Moreover, the *symplectic group* $\mathrm{Sp}(2)$—the set of all linear transformations of $\mathbb{R}^2$ preserving ω—coincides with $\mathrm{SL}_2(\mathbb{R})$.

Now we define the main object of our consideration.

Definition

An element of the n-dimensional *Heisenberg group* $\mathbb{H}^n$ is $(s, x, y) \in \mathbb{R}^{2n+1}$, where $s \in \mathbb{R}$ and x, $y \in \mathbb{R}^n$. The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) \cdot (s', x', y') = (s + s' + \tfrac{1}{2}\omega(x, y; x', y'), x + x', y + y'), \tag{4}$$

where ω the symplectic form.

For the Heisenberg group $\mathbb{H}^n$, check that:

1. The unit is $(0, 0, 0)$ and the inverse $(s, x, y)^{-1} = (-s, -x, -y)$.
2. It is a non-commutative, the *centre* of $\mathbb{H}^n$ is:
$$Z = \{(s, 0, 0) \in \mathbb{H}^n,\ s \in \mathbb{R}\}. \tag{5}$$
3. The group law is continuous, so we have a Lie group.

Alternative group laws I

for the Heisenberg group

1. Introduce complexified coordinates (s, z) on $\mathbb{H}^1$ with $z = x + iy$. Then the group law can be written as:

$$(s, z) \cdot (s', z') = (s + s' + \tfrac{1}{2}\Im(z'\bar{z}), z + z').$$

2. Show that the set $\mathbb{R}^3$ with the group law

$$(s, x, y) * (s', x', y') = (s + s' + xy', x + x', y + y') \tag{6}$$

is isomorphic to the Heisenberg group $\mathbb{H}^1$. It is called the *polarised Heisenberg group*. HINT: Use the explicit form of the homomorphism $(s, x, y) \mapsto (s + \frac{1}{2}xy, x, y)$.$\diamond$

Alternative group laws II

for the Heisenberg group

3. Define the map $\phi : \mathbb{H}^1 \to M_3(\mathbb{R})$ by

$$\phi(s, x, y) = \begin{pmatrix} 1 & x & s + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

   This is a group homomorphism from $\mathbb{H}^1$ to the group of 3×3 matrices with the unit determinant and the matrix multiplication as the group operation. Write also a group homomorphism from the polarised Heisenberg group to $M_3(\mathbb{R})$.

4. Expand the above items from this Exercise to $\mathbb{H}^n$.

The Weyl algebra

The key idea of analysis is a linearization of complicated object in small neighbourhoods. Applied to Lie groups it leads to the Lie algebras. The Lie algebra of the Heisenberg group $\mathfrak{h}_1$ is also called *Weyl algebra*. From the general theory we know, that $\mathfrak{h}_1$ is a three-dimensional real vector space, thus, it can be identified as a set with the group $\mathbb{H}^1 \sim \mathbb{R}^3$ itself.

There are several standard possibilities to realise $\mathfrak{h}_1$:

1. Generators X of one-parameter subgroups: $x(t) = \exp(Xt)$, $t \in \mathbb{R}$.
2. Tangent vectors to the group at the group unit.
3. Invariant first-order vector fields (differential operators) on the group.

There is the important exponential map between a Lie algebra and respective Lie group. The exponent function can be defined in any topological algebra as the sum of the following series:

$$\exp(tX) = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}.$$

Generators of subgroups
and the exponential map

1. Matrices from (7) are created by the following exponential map:

$$\exp\begin{pmatrix} 0 & x & s \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x & s+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

Thus $\mathfrak{h}_1$ isomorphic to the vector space of matrices in the left-hand side. We can define the explicit identification $\exp : \mathfrak{h}_1 \to \mathbb{H}^1$ by (8), which is also known as the exponential coordinates on $\mathbb{H}^1$.

2. Define the basis of $\mathfrak{h}_1$:

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

Write the one-parameter subgroups of $\mathbb{H}^1$ generated by S, X and Y.

Invariant vector fields

A (continuous) *one-parameter subgroup* is a continuous group homomorphism F from $(\mathbb{R},+)$ to $\mathbb{H}^n$:

$$F(t+t') = F(t)\cdot F(t').$$

We can calculate the left and right derived action at any point $g \in \mathbb{H}^n$:

$$\left.\frac{d(F(t)\cdot g)}{dt}\right|_{t=0} \quad \text{and} \quad \left.\frac{d(g\cdot F(t))}{dt}\right|_{t=0}. \tag{10}$$

1. Check that the following vector fields on $\mathbb{H}^1$ are left (right) invariant:

$$S^{l(r)} = \pm\partial_s, \quad X^{l(r)} = \pm\partial_x - \tfrac{1}{2}y\partial_s, \quad Y^{l(r)} = \pm\partial_y + \tfrac{1}{2}x\partial_s. \tag{11}$$

   Show, that they are linearly independent and, thus, are bases of the Lie algebra $\mathfrak{h}_1$ (in two different realizations).

2. Find one-parameter groups of right (left) shifts on $\mathbb{H}^1$ generated by these vector fields.

Commutator I

The principal operation on a Lie algebra, besides the linear structure, is the Lie bracket—a bi-linear, anti-symmetric form with the Jacoby identity:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

In the above exercises, as in any algebra, we can define the Lie bracket as the commutator $[A, B] = AB - BA$: for matrices and vector fields through the corresponding algebraic operations in these algebras.

1. Check that bases from (9) and (11) satisfy the Heisenberg *commutator relation*

   $$[X^{l(r)}, Y^{l(r)}] = S^{l(r)} \tag{12}$$

   and all other commutators vanishing. More generally:

   $$[A, A'] = \omega(x, y; x', y')S, \qquad \text{where } A^{(\prime)} = s^{(\prime)}S + x^{(\prime)}X + y^{(\prime)}Y, \tag{13}$$

   and ω is the symplectic form.

Commutator II

2. Show that any second (and, thus, any higher) commutator $[[A, B], C]$ on $\mathfrak{h}_1$ vanishes. This property can be stated as "the Heisenberg group is a step 2 nilpotent Lie group".
3. Check the formula

$$\exp(A)\exp(B) = \exp(A + B + \tfrac{1}{2}[A, B]), \qquad \text{where } A, B \in \mathfrak{h}_1. \tag{14}$$

   The formula is also true for any step 2 nilpotent Lie group and is a particular case of the Baker–Campbell–Hausdorff formula. HINT: In the case of $\mathbb{H}^1$ you can use the explicit form of the exponential map (8).$\diamond$
4. Define the vector space decomposition

$$\mathfrak{h}_1 = V_0 \oplus V_1, \qquad \text{such that } V_0 = [V_1, V_1]. \tag{15}$$

Erlangen programme suggest investigate invariants under group action. This recipe can be applied recursively to groups themselves. Transformations of a group which preserve its structure are called *group automorphisms*. The following are automorphisms of $\mathbb{H}^1$:

1. *Inner automorphisms* or *conjugation* with $(s, x, y) \in \mathbb{H}^1$:
$$\begin{aligned}(s', x', y') &\mapsto (s, x, y) \cdot (s', x', y') \cdot (s, x, y)^{-1} \\ &= (s' + \omega(x, y; x', y'), x', y') = (s' + xy' - x'y, x', y').\end{aligned} \tag{16}$$
2. *Symplectic maps* $(s, x, y) \mapsto (s, \tilde{x}, \tilde{y})$, where $\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$ with A from the symplectic group $\mathrm{Sp}(2) \sim \mathrm{SL}_2(\mathbb{R})$, see Exercise 3.
3. *Dilations*: $(s, x, y) \mapsto (r^2 s, rx, ry)$ for a positive real r.
4. *Inversion*: $(s, x, y) \mapsto (-s, y, x)$.

The last three types of transformations are *outer automorphisms*.

The group of all automorphisms

of the Heisenberg group

Show that

1. Automorphism groups of $\mathbb{H}^1$ and $\mathfrak{h}_1$ coincide as groups of maps of $\mathbb{R}^3$ onto itself. HINT: Use the exponent map and the relation (14). The crucial step is a demonstration that any automorphism of $\mathbb{H}^1$ is a linear map of $\mathbb{R}^3$.$\diamond$
2. All transforms from the previous slide, viewed as automorphisms of $\mathfrak{h}_1$, preserve the decomposition (15).
3. Every automorphism of $\mathbb{H}^1$ can be written uniquely as composition of a symplectic map, an inner automorphism, a dilation and a power $(\mod 2)$ of the inversion from the previous slide. HINT: Any automorphism is a linear map (by the previous item) of $\mathbb{R}^3$ which maps the centre Z to itself. Thus it shall have the form $(s, x, y) \mapsto (cs + ax + by, T(x, y))$, where a, b and c are real and T is a linear map of $\mathbb{R}^2$.$\diamond$

The Scrödinger group

For future use we will need $\widetilde{\mathrm{Sp}}(2)$ which is the double cover of $\mathrm{Sp}(2)$. We can build the semidirect product $\mathsf{G} = \mathbb{H}^1 \rtimes \widetilde{\mathrm{Sp}}(2)$ with the standard group law for semidirect products:

$$(\mathsf{h}, \mathsf{g}) * (\mathsf{h}', \mathsf{g}') = (\mathsf{h} * \mathsf{g}(\mathsf{h}'), \mathsf{g} * \mathsf{g}'), \tag{17}$$

where h, $\mathsf{h}' \in \mathbb{H}^1$, $\mathsf{g}, \mathsf{g}' \in \widetilde{\mathrm{Sp}}(2)$. Here the stars denote the respective group operations while the action $\mathsf{g}(\mathsf{h}')$ is defined as the composition of the projection map $\widetilde{\mathrm{Sp}}(2) \to \mathrm{Sp}(2)$ and the action (3).
This group is sometimes called the *Schrödinger group* and it is known as the maximal kinematical invariance group of both the free Schrödinger equation and the quantum harmonic oscillator. This group is of interest not only in quantum mechanics but also in optics.

Subgroups
and Homogeneous Spaces

Let G be a group and H be its closed subgroup.
The *homogeneous space* G/H from the equivalence relation: $\mathsf{g}' \sim \mathsf{g}$ iff $\mathsf{g}' = \mathsf{g}\mathsf{h}$, $\mathsf{h} \in \mathsf{H}$. The *natural projection* $\mathsf{p} : \mathsf{G} \rightarrow \mathsf{G}/\mathsf{H}$ puts $\mathsf{g} \in \mathsf{G}$ into its equivalence class.
A continuous section $\mathsf{s} : \mathsf{G}/\mathsf{H} \rightarrow \mathsf{G}$ is a right inverse of p, i.e. $\mathsf{p} \circ \mathsf{s}$ is an identity map on G/H. Then the *left action* of G on itself $\Lambda(\mathsf{g}) : \mathsf{g}' \mapsto \mathsf{g}^{-1} * \mathsf{g}'$ generates the action on G/H:

$$\mathsf{g} : \mathsf{x} \mapsto \mathsf{p}(\mathsf{g}^{-1} * \mathsf{s}(\mathsf{x})), \quad \text{or graphically} \quad \begin{array}{ccc} \mathsf{G} & \xrightarrow{\mathsf{g}^{-1}*} & \mathsf{G} \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} \\ \mathsf{G}/\mathsf{H} & \xrightarrow{\mathsf{g}\cdot} & \mathsf{G}/\mathsf{H} \end{array}$$

We want to classify up to certain equivalences all possible $\mathbb{H}^1$–homogeneous spaces. According to the diagram we will look for *continuous* subgroups of $\mathbb{H}^1$.

One-dimensional continuous subgroups of $\mathbb{H}^1$ can be classified up to group automorphism. Two one-dimensional subgroups of $\mathbb{H}^1$ are the centre Z (5) and

$$H_x = \{(0, t, 0) \in \mathbb{H}^1, \ t \in \mathbb{R}\}. \tag{18}$$

Show that:

1. There is no an automorphism which maps Z to H_x.
2. For any one-parameter continuous subgroup of $\mathbb{H}^1$ there is an automorphism which maps it either to Z or H_x.
3. The classification of one-parameter subgroups can be based on their infinitesimal generators from the Weyl algebra.

Next, we wish to describe the respective homogeneous spaces and actions of $\mathbb{H}^1$ on them. Check that:

1. The $\mathbb{H}^1$-action on $\mathbb{H}^1/Z$ is:

$$(s, x, y) : (x', y') \mapsto (x + x', y + y'). \tag{19}$$

   HINT: Use the following maps: $\mathsf{p} : (s', x', y') \mapsto (x', y')$, $\mathsf{s} : (x', y') \mapsto (0, x', y')$.$\diamond$

2. The $\mathbb{H}^1$-action on $\mathbb{H}^1/H_x$ is:

$$(s, x, y) : (s', y') \mapsto (s + s' + xy' + \tfrac{1}{2}xy, y + y'). \tag{20}$$

   HINT: Use the following maps: $\mathsf{p} : (s', x', y') \mapsto (s' + \frac{1}{2}x'y', y')$, $\mathsf{s} : (s', y') \mapsto (s', 0, y')$.$\diamond$

3. Calculate the derived action similar to (10).

2D Subgroups of $\mathbb{H}^1$

and 1D homogeneous spaces

The classification of two-dimensional subgroups is as follows:
Show that

1. For any two-dimensional continuous subgroup of $\mathbb{H}^1$ there is an automorphism of $\mathbb{H}^1$ which maps the subgroup to

$$H'_x = \{(s, 0, y) \in \mathbb{H}^1,\ s, y \in \mathbb{R}\}.$$

2. $\mathbb{H}^1$-action on $\mathbb{H}^1/H'_x$ is

$$(s, x, y) : x' \mapsto x + x'. \qquad (21)$$

   HINT: Use the maps $p : (s', x', y') \mapsto x'$ and $s : x' \mapsto (0, x', 0)$.◇

3. Calculate the derived action similar to (10).

Actions (19) and (21) are simple shifts. Nevertheless, the associated representations of the Heisenberg group will be much more interesting.

Group Representations

Definition (traditional)

A (linear) *representation* ρ of a group G is a group homomorphism $\rho : G \to B(V)$ from G to (bounded) linear operators on a space V:

$$\rho(gg') = \rho(g)\rho(g').$$

Informally: A representation of G is an introduction of an operation of addition on G, which is compatible with group multiplication.

Example

The following are group representations:

1. Let $G = (\mathbb{R}, +)$, $V = \mathbb{C}$, $\rho(x) = e^{iax}$, $a \in \mathbb{R}$. It is a 1D-representation called a *character*.
2. Let $G = (\mathbb{R}, +)$, $V = L_2(\mathbb{R})$, representation by shifts: $[\rho(x)f](t) = f(x + t)$ is infinite-dimensional.
3. For any group G shifts $f(g') \mapsto f(g^{-1}g')$ and $f(g') \mapsto f(g'g)$ are the left and right *regular* representations.

Continuous Representations of Topological Groups

A representation is a map which respects the group structure. If we have a topological group, it is natural to consider representations respecting topology as well, that is representation which are continuous in some topology. It is most common (and convenient!) to use the following type.

Definition

A representation ρ of G in a vector space V is *strong continuous* if for any convergent sequence $(g_n) \to g \in G$ and for any $x \in V$ we have $\|\rho(g_n)x - \rho(g)x\| \to 0$.

Exercise

Which representations from the previous Example 1 are strongly continuous in a suitable topology?

From now, we consider strongly continuous representations only.

Decomposition of Representations

Definition

A subspace $U \subset V$ is called *invariant* if $\rho(g)U \subset U$ for all $g \in G$. We can always consider a restriction of ρ to any its invariant subspace. Such a restriction is called *subrepresentation*.

Definition

A representation is *irreducible* if the only closed invariant subspaces are trivial (the whole V and $\{0\}$). Otherwise it is *reducible*.

The regular representation of $(\mathbb{R}, +)$ on $V = L_2(\mathbb{R})$ by shifts has closed invariant subspaces, e.g. the Hardy space—space of all functions having an analytic extension to the upper half-plane. So it is reducible. A character (and any 1D-representation) is an irreducible representation.

Definition

A representation is *decomposable* if $V = V_1 \oplus V_2$, where V_1 and V_2 are invariant.

Unitary Representations

The representation theory is much simpler if representing operators belong to a nice class.

Definition

A strongly continuous representation ρ of G in V is *unitary* if V is a Hilbert space and all $\rho(g)$, $g \in G$ are unitary operators.

Exercise

Define Hilbert spaces such that representations from the previous Example 1 becomes unitary.

One of the important properties of unitary representations is complete reducibility. Namely, a representation can be reducible but indecomposable. However, any reducible unitary representation is decomposable: for any closed invariant subspace, the orthogonal complement is again a closed invariant subspace.
We will consider unitary representations of $\mathbb{H}^n$ only.

Induced Representations

Let G be a group, H its closed subgroup, χ be a linear representation of H in a space V. The set of V-valued functions with the property

$$F(gh) = \chi(h)F(g),$$

is invariant under left shifts. The restriction of the left regular representation to this space is called an *induced representation*. Consider the *lifting* $\mathcal{L}_\chi$ of $f(x)$, $x \in X = G/H$ to $F(g)$:

$$F(g) = [\mathcal{L}_\chi f](g) = \chi(h)f(p(g)), \quad p : G \to X, \quad g = s(x)h, \quad x = p(g),$$

The map transforms the left regular representation on G to the following action:

$$[\rho(g)f](x) = \chi(r(g^{-1} * s(x)))f(g \cdot x), \qquad \text{from} \qquad \begin{array}{ccc} G & \xrightarrow{g^{-1}*} & G \\ s \uparrow\downarrow p & & s \uparrow\downarrow p \\ G/H & \xrightarrow{g\cdot} & G/H \end{array},$$

where $r : G \to H$ by $r(g) = s(x)^{-1}g$ for $x = p(g)$.

1. For $H = Z$ the map $r : \mathbb{H}^1 \to Z$ is $r(s,x,y) = (s,0,0)$. For the character $\chi_\hbar(s,0,0) = e^{2\pi i \hbar s}$, the representation of $\mathbb{H}^1$ on $L_2(\mathbb{R}^2)$ is, cf. (19):

$$[\rho_\hbar(s,x,y)f](x',y') = e^{2\pi i\hbar(-s-\frac{1}{2}\omega(x,y;x'y'))}f(x'-x,y'-y). \quad (22)$$

This is the *Fock–Segal–Bargmann (FSB) representation.*

2. For $H = H_x$ the map $r(s,x,y) = (0,x,0)$. For the character $\chi(0,x,0) = e^{2\pi i\hbar x}$, the representation $\mathbb{H}^1$ on $L_2(\mathbb{R}^2)$ is, cf. (20):

$$[\rho_\hbar(s,x,y)f](s',y') = e^{-2\pi i\hbar x}f(s'-s-xy'+\frac{1}{2}xy, y'-y). \quad (23)$$

For $H = H'_x = \{(s, 0, y) \in \mathbb{H}^1\}$ the map $r : \mathbb{H}^1 \to H'_x$ is $r(s, x, y) = (s - \frac{1}{2}xy, 0, y)$.
For the character $\chi_\hbar(s, 0, y) = e^{2\pi i(\hbar s + qy)}$, the representation of $\mathbb{H}^1$ on $L_2(\mathbb{R}^1)$ is, cf. (21):

$$[\rho_\hbar(s, x, y)f](x') = \exp(2\pi i(\hbar(-s + yx' - \tfrac{1}{2}xy) - qy))\, f(x' - x). \quad (24)$$

For $q = 0$, this a key to the *Schrödinger representation* of the Heisenberg group.
We built many representations of $\mathbb{H}^1$, are they essentially different?
We will show that any two above representations $\rho_\hbar$ and $\rho'_\hbar$ with the same value of the parameter $\hbar$ are unitary equivalent, namely there is a unitary operator U intertwining them:

$$\rho_\hbar(g)U = U\rho'_\hbar(g), \qquad \text{for all } g \in G.$$

Adjoint Representation

for a Matrix Group

G—a *matrix group*, i.e. subgroup and a smooth submanifold of $GL(n, \mathbb{R})$.

$\mathfrak{g} = \mathrm{Lie}(G)$—the Lie algebra, the tangent space $T_e(G)$ to G at the unit e.

$A(g) : x \mapsto gxg^{-1}$—$G$-action on itself by inner automorphisms. It fixes the group unit e and thus generates a linear transformation of the tangent space at e, which is identified with $\mathfrak{g}$.

$A_*(g) : \mathfrak{g} \mapsto \mathfrak{g}$—the above derived map which is usually denoted by $Ad(g)$.

$g \mapsto Ad(g)$—is called the *adjoint representation* of G.

Luckily, this construction can greatly simplified for matrix groups: the adjoint representation is *matrix conjugation*:

$$Ad(g)B = gBg^{-1}, \qquad \text{where} \quad B \in \mathfrak{g}, \quad g \in G.$$

Co-Adjoint Representation

Dual to Adjoint One

$\mathfrak{g}^*$—dual space to the Lie algebra $\mathfrak{g}$.

$\langle A, B \rangle = \operatorname{tr}(AB)$—a bilinear form on $\mathrm{Mat}_n(\mathbb{R})$ invariant under conjugation.

$\mathfrak{g}^\perp$—the orthogonal complement of $\mathfrak{g}^*$ in $\mathrm{Mat}_n(\mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle$.

$\mathrm{Mat}_n(\mathbb{R})/\mathfrak{g}^\perp$—model for $\mathfrak{g}^*$.

p— the projection of $\mathrm{Mat}_n(\mathbb{R})$ on $\mathfrak{g}^*$ parallel to $\mathfrak{g}^\perp$.

Then the *co-adjoint representation* K, which is dual to the adjoint representation defined above, can be written in the simple form

$$K(g) : F \mapsto p(gFg^{-1}), \qquad \text{where} \quad F \in \mathfrak{g}^*.$$

Under the co-adjoint representation $\mathfrak{g}^*$ is split into a family of disjoint orbits, giving the name *orbit method* by Kirillov.

Co-Adjoint Representation

For the Heisenberg group

Realising $\mathbb{H}^1$ as a matrix group, we calculate the matrix conjugation:

$$g = \begin{pmatrix} 1 & x & s+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}^1, \quad B = \begin{pmatrix} 0 & x' & s' \\ 0 & 0 & y' \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}^1,$$

$$\mathrm{Ad}(g)B = \begin{pmatrix} 0 & x' & -x'y+xy'+s' \\ 0 & 0 & y' \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} s'' \\ x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & -y & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s' \\ x' \\ y' \end{pmatrix}$$

We introduce coordinates $(\hbar, q, p)$ in $\mathfrak{h}^*_n \sim \mathbb{R}^{2n+1}$ in bi-orthonormal coordinates to the exponential ones (s, x, y) on $\mathfrak{h}^n$. Then the co-adjoint representation $\mathrm{Ad}^* : \mathfrak{h}^*_n \to \mathfrak{h}^*_n$ becomes:

$$\mathrm{Ad}^*(s,x,y) : (\hbar, q, p) \mapsto (\hbar, q - \hbar y, p + \hbar x), \quad \text{where } (s,x,y) \in \mathbb{H}^n \tag{25}$$

Note, that every $(0, q, p)$ is fixed. Also all hyperplanes $\hbar = \text{const} \neq 0$ are orbits and the action on them is similar to (19).

Orbit Space

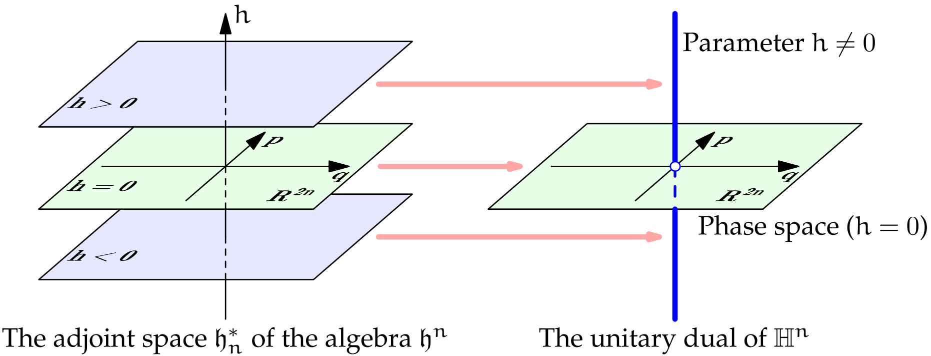

Figure: The structure of unitary dual to $\mathbb{H}^n$ from the method of orbits. The space $\mathfrak{h}_n^*$ is sliced into "horizontal" hyperplanes. Planes with $\hbar \neq 0$ form single orbits and correspond to different classes of UIR. The plane $\hbar = 0$ is a family of one-point orbits $(0, q, p)$, which produce one-dimensional representations. The topology on the dual object is the factor topology inherited from the $\mathfrak{h}_n^*$.

Physical Units

shall not be neglected

Let M be a unit of mass, L—of length, T—of time. We adopt the following

Convention

1. Only physical quantities of the *same dimension* can be added or subtracted. However, any quantities can be multiplied/divided.
2. Therefore, mathematical functions, e.g. $\exp(u) = 1 + u + u^2/2! + \ldots$ or $\sin(u)$, can be constructed out of a dimensionless number u only. Thus, Fourier dual variables, say x and q, should posses reciprocal dimensions to enter the expression e^{ixq}.
3. For physical reasons being seen later, we assign to x and y components of (s, x, y) physical units $1/\mathsf{L}$ and $\mathsf{T}/(\mathsf{LM})$ respectively.

Consequently, the parameter s should be measured in $\mathsf{T}/(\mathsf{L}^2\mathsf{M})$—the product of units of x and y. The coordinates $\hbar$, q, p should have units of an *action* ML^2/T, *coordinates* L, and *momenta* LM/T, respectively.

We now build induced representations generated by the coadjoint orbits. Starting from the action (25) on an orbit $\hbar \neq 0$ and the character $e^{-2\pi i\hbar s}$ of the centre we obtain the representation:

$$\rho_\hbar(s,x,y): f(q,p) \mapsto e^{-\pi i(2\hbar s+qx+py)} f\left(q-\hbar y, p+\hbar x\right). \qquad (26)$$

The same formula is obtained if we use the Fourier transform $(x',y') \to (q,p)$ for the representation (22). Note that the representation obeys our agreement on physical units, if (q,p) is treated as a point of the phase space.

Similarly, we can apply the Fourier transform $x' \to q$ for the representation (24) and obtain the *Schrödinger representation* :

$$[\rho_\hbar(s,x,y)f](q) = e^{-\pi i\hbar(2s+xy)-2\pi i x q}\, f(q+\hbar y). \qquad (27)$$

The variable q is treated as the coordinate on the configurational space of a particle.

Derivation of Representations

Let ρ be a representation of a Lie group G with the Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ and real t we have $\exp(tX) \in G$.
For any representation ρ of G in a space V we obtain one-parameter semigroup of operators $\rho(\exp(Xt))$ on V. We can try to calculate its generator:

$$d\rho^X := \left.\frac{d\,\rho(\exp(Xt))}{dt}\right|_{t=0}. \tag{28}$$

Even for a bounded representation ρ the above operator may be unbounded and we need to define its domain as a proper subspace $U \subset V$. In this way obtain the *derived representation* of the Lie algebra $\mathfrak{g}$.

Example

1. Let $G = (\mathbb{R}, +)$, $V = \mathbb{C}$, $\rho(x) = e^{iax}$, $a \in \mathbb{R}$. The derived representation is $d\rho^T = iaT$ for $T \in \mathfrak{r} \sim \mathbb{R}$.
2. Let $G = (\mathbb{R}, +)$, $V = L_2(\mathbb{R})$, $[\rho(x)f](t) = f(x + t)$ then $d\rho^T = T\frac{d}{dx}$. As the domain we can take the Schwartz space $S(\mathbb{R})$.

Invariant measure

Definition

Let G be a Lie group, a measure μ is left (right) invariant (aka the Haar measure if $\mu(g \cdot A) = \mu(A)$ for any $A \subset G$ and $g \in G$.

If there is measure simultaneously left and right invariant then the group is called *unimodular*.

Example

1. Any commutative or compact group is unimodular;
2. The Heisenberg group is unimodular with the Haar measure $ds\,dx\,dy$ coinciding with the Lebesgue measure $\mathbb{R}^{2n+1}$.
3. The $ax + b$ group (affine maps of the real line) is *not* unimodular.

For a left Haar measure μ, the left regular representation $[\Lambda(g)f](g') = f(g^{-1}g')$ is an isometry on $L_p(G, \mu)$:

$$\int_G \left|f(g^{-1}g')\right|^p \, d\mu(g') = \int_G |f(g')|^p \, d\mu(g').$$

Let G be a Lie group with a left Haar measure μ, for a function $k \in L_1(G, \mu)$ we can define the operator K on bounded functions with compact supports:

$$[Kf](g') = (k * f)(g') = \int_G k(g) f(g^{-1} \cdot g') \, d\mu(g). \qquad (29)$$

It is called the *convolution* operator and k is called the *kernel* of the convolution. For $k \in L_1(G, \mu)$, K is a bounded operator on $L_2(G, \mu)$. The composition $K_1 K_2$ of two such operators with kernels k_1 and k_2 is again a convolution with the kernel $k_1 * k_2$—$L_1(G, \mu)$ is a *convolution algebra*.

Similarly, let ρ be bounded representation of G in a normed space V and $k \in L_1$, define the operator

$$\rho(k) = \int_G k(g) \rho(g) \, d\mu(g). \qquad (30)$$

This is a representation of the $L_1(G, \mu)$ algebra:

$$\rho(k_1 * k_2) = \rho(k_1) \rho(k_2).$$

In 1926, Dirac discussed the idea that quantum mechanics can be obtained from classical one through a change in the only rule:

> *. . . there is one basic assumption of the classical theory which is false, and that if this assumption were removed and replaced by something more general, the whole of atomic theory would follow quite naturally. Until quite recently, however, one has had no idea of what this assumption could be.*

By Dirac, such a condition is provided by the Heisenberg commutation relation of coordinate and momentum:

$$q_r p_r - p_r q_r = i\hbar, \tag{31}$$

i.e. a representation (??) of the Heisenberg–Weyl algebra $\mathfrak{h}_n$:

> *The new mechanics of the atom introduced by Heisenberg may be based on the assumption that the variables that describe a dynamical system do not obey the commutative law of multiplication, but satisfy instead certain quantum conditions.*

Non-Essential Noncommutativity

Noncommutativity of observables is not an essential prerequisite for quantum mechanics: there are constructions of quantum theory which do not relay on it, e.g. the Feynman path integral. In the popular lectures [1] Feynman managed to tell the fundamental features of quantum electrodynamics without any reference to (non-)commutativity. But what is the mathematical source of quantum theory if noncommutativity is not? The vivid presentation in [1] uses stopwatch with a single hand to present the *phase* for a path $x(t)$ between two points in the configuration space. The mathematical expression for the path's phase is [2, (2-15)]:

$$\phi[x(t)] = \text{const} \cdot e^{(i/\hbar)S[x(t)]}, \tag{32}$$

where $S[x(t)]$ is the *classic action* along the path $x(t)$. Summing up contributions (32) along all paths between two points a and b we obtain the amplitude $K(a, b)$. This amplitude presents very accurate description of many quantum phenomena. Therefore, expression (32) is also a strong contestant for the rôle of the cornerstone of quantum theory.

Path Integral Illustration

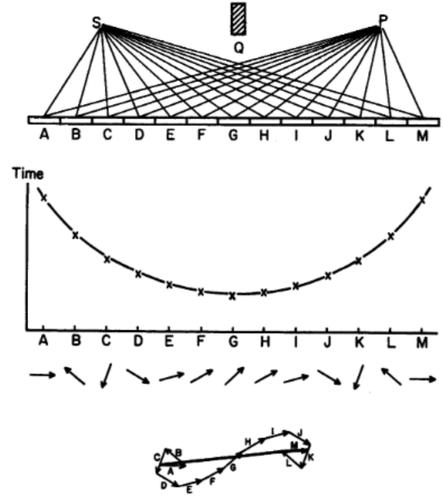

FIGURE 24. *Each path the light could go (in this simplified situation) is shown at the top, with a point on the graph below it showing the time it takes a photon to go from the source to that point on the mirror, and then to the photomultiplier. Below the graph is the direction of each arrow, and at the*

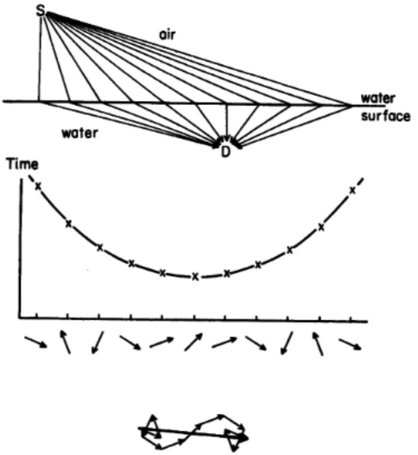

FIGURE 29. *Quantum theory says that light can go from a source in air to a detector in water in many ways. If the problem is simplified as in the case of the mirror, a graph showing the timing of each path can be drawn, with the direction of each arrow below it. Once again, the major contribution toward*

The non-zero Planck constant and QM

Is there anything common in two "principal" identities (31) and (32)

$$q_r p_r - p_r q_r = i\hbar \qquad \text{and} \qquad \phi[x(t)] = \text{const} \cdot e^{(i/\hbar)S[x(t)]} \ ?$$

The only two common elements are (in order of "significance"):

1. The non-zero Planck constant $\hbar$.
2. The imaginary unit i.

The Planck constant was the first manifestation of quantum (discrete) behaviour and it is at the heart of the whole theory. In contrast, classical mechanics is oftenly obtained as a semiclassical limit $\hbar \to 0$, see also Fig. 1. Thus, the non-zero Planck constant looks like a clear marker of quantum world in its opposition to the classical one.

The complex imaginary unit is also a mandatory element of quantum mechanics in all its possible formulations. E.g. the popular lectures [1] managed to avoid any noncommutativity issues but did mention complex numbers both explicitly (see the Index there) and implicitly (as rotations of the hand of a stopwatch). However, it is a common perception that complex numbers are useful but manly technical tool in quantum theory.

For a Hamiltonian $H(q,p)$ we can integrate the representation $\rho_\hbar$ with the Fourier transform $\hat{H}(x,y)$ of $H(q,p)$, see (30):

$$\tilde{H} = \int_{\mathbb{R}^2} \hat{H}(x,y)\, \rho_\hbar(0,x,y)\, dx\, dy$$

and obtain (possibly unbounded) PDO $\tilde{H}$ (??) on $L_2(\mathbb{R}^2)$. This assignment of the operator $\tilde{H}$ (quantum observable) to a function $H(q,p)$ (classical observable) is known as the *Weyl quantization* or a *Weyl calculus* of PDO. The Hamiltonian $\tilde{H}$ defines the dynamics of a quantum observable $\tilde{k}$ by the *Heisenberg equation*:

$$ih\frac{d\tilde{k}}{dt} = \tilde{H}\tilde{k} - \tilde{k}\tilde{H}, \tag{33}$$

where $[\tilde{H},\tilde{k}] = \tilde{H}\tilde{k} - \tilde{k}\tilde{H}$ is the commutator of the observable $\tilde{k}$ and the Hamilton $\tilde{H}$.

Instead of the imaginary unit with the property $i^2 = -1$ we will use the nilpotent unit ε such that $\varepsilon^2 = 0$. The *dual numbers* generated by nilpotent unit were already known for there connections with Galilean relativity—the fundamental symmetry of classical mechanics—thus its appearance in our discussion shall not be very surprising
Consider a four-dimensional algebra $\mathfrak{C}$ spanned by 1, i, ε and $i\varepsilon$. We can define the following representation $\rho_{\varepsilon h}$ of the Heisenberg group in a space of $\mathfrak{C}$-valued smooth functions [6–8]:

$$\rho_{\varepsilon h}(s,x,y):\ f(q,p) \mapsto e^{-2\pi i(xq+yp)}\left(f(q,p)+\varepsilon h\left(sf(q,p)+\frac{y}{4\pi i}f'_q(q,p)-\frac{x}{4\pi i}f'_p(q,p)\right)\right). \tag{34}$$

A simple calculation shows the representation property $\rho_{\varepsilon h}(s,x,y)\rho_{\varepsilon h}(s',x',y') = \rho_{\varepsilon h}((s,x,y)*(s',x',y'))$ for the multiplication (4) on $\mathbb{H}^1$. Since this is not a unitary representation in a complex-valued Hilbert space its existence does not contradict the Stone–von Neumann theorem ??.

Comparison of Two Representations

Recall the induced representation (26) obtained from the orbit method:

$$\rho_\hbar(s,x,y): f(q,p) \mapsto e^{-\pi i(2\hbar s+qx+py)} f\left(q-\hbar y, p+\hbar x\right). \qquad (35)$$

To highlight the similarity, we re-write (34) as:

$$\rho_\hbar(s,x,y): f(q,p) \mapsto e^{-2\pi(\varepsilon\hbar s+i(qx+py))} f\left(q-\frac{i\hbar}{2}\varepsilon y, p+\frac{i\hbar}{2}\varepsilon x\right). \qquad (36)$$

Here, for a differentiable function k of a real variable t, the expression $k(t+\varepsilon a)$ is understood as $k(t)+\varepsilon a k'(t)$, where $a \in \mathbb{C}$ is a constant. For a real-analytic function k this follows from its Taylor's expansion.

Both representations (35) and (36) are *noncommutative* and act on the phase space. The important distinction is:

- The representation (35) is induced by the *complex-valued* unitary character $\rho_\hbar(s,0,0)=e^{2\pi i\hbar s}$ of the centre Z of $\mathbb{H}^1$.
- The representation (36) is similarly induced by the *dual number-valued* character $\rho_{\varepsilon h}(s,0,0)=e^{\varepsilon hs}=1+\varepsilon hs$ of the centre Z of $\mathbb{H}^1$. Here dual numbers are the associative and commutative two-dimensional algebra spanned by 1 and ε.

Dual Number Representations

Infinitesimal Form

The infinitesimal generators of one-parameter subgroups $\rho_{\varepsilon h}(0,x,0)$ and $\rho_{\varepsilon h}(0,0,y)$ in (34) are

$$d\rho^X_{\varepsilon h} = -2\pi i q - \frac{\varepsilon h}{4\pi i}\partial_p \quad \text{and} \quad d\rho^Y_{\varepsilon h} = -2\pi i p + \frac{\varepsilon h}{4\pi i}\partial_q,$$

respectively. We calculate their commutator:

$$d\rho^X_{\varepsilon h} \cdot d\rho^Y_{\varepsilon h} - d\rho^Y_{\varepsilon h} \cdot d\rho^X_{\varepsilon h} = \varepsilon h. \tag{37}$$

It is similar to the Heisenberg relation (31): the commutator is non-zero and is proportional to the Planck constant. The only difference is the replacement of the imaginary unit by the nilpotent one. The radical nature of this change becomes clear if we integrate this representation with the Fourier transform $\hat{H}(x,y)$ of a Hamiltonian function $H(q,p)$:

$$\mathring{H} = \int_{\mathbb{R}^{2n}} \hat{H}(x,y)\,\rho_{\varepsilon h}(0,x,y)\,dx\,dy = H + \frac{\varepsilon h}{2}\left(\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}\right). \tag{38}$$

This is a first order differential operator on the phase space.

The Hamilton Equation

The differential operator $\mathring{H}$ (38) generates a dynamics of a classical observable k—a smooth real-valued function on the phase space—through the equation isomorphic to the Heisenberg equation (33):

$$\varepsilon h \frac{d\mathring{k}}{dt} = \mathring{H}\mathring{k} - \mathring{k}\mathring{H}.$$

Making a substitution from (38) and using the identity $\varepsilon^2 = 0$ we obtain:

$$\frac{dk}{dt} = \frac{\partial H}{\partial p}\frac{\partial k}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial k}{\partial p}. \tag{39}$$

This is, of course, the *Hamilton equation* of classical mechanics based on the *Poisson bracket*:

$$\{\mathring{H}, \mathring{k}\} = \frac{\partial H}{\partial p}\frac{\partial k}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial k}{\partial p}.$$

Quantum/Classical Mechanics

=Complex/Dual representations $\mathbb{H}^n$

1. Noncommutativity is not a crucial prerequisite for quantum theory, it can be obtained as a consequence of other fundamental assumptions.
2. Noncommutativity is not a distinguished feature of quantum theory, there are noncommutative formulations of classical mechanics as well.
3. The non-zero Planck constant is compatible with classical mechanics. Thus, there is no a necessity to consider the semiclassical limit $\hbar \to 0$, where the *constant* has to tend to zero.
4. There is no a necessity to request that physical observables form an algebra. Quantization can be performed by the Weyl recipe, which requires only a structure of a linear space in the collection of all observables with the same physical dimensionality.
5. The imaginary unit in (31) is ultimately responsible for most of quantum effects. Classical mechanics can be obtained from the similar commutator relation (37) using the nilpotent unit $\varepsilon^2 = 0$.

Bibliography

[1] R.P. Feynman. *QED: the strange theory of light and matter*. Penguin Press Science Series. Penguin, 1990. ↑37, 39

[2] R.P. Feynman and A.R. Hibbs. *Quantum mechanics and path integral*. McGraw-Hill Book Company, New York, 1965. ↑37

[3] Gerald B. Folland. *Harmonic analysis in phase space*. Annals of Mathematics Studies, vol. 122. Princeton University Press, Princeton, NJ, 1989. ↑

[4] Roger Howe. On the role of the Heisenberg group in harmonic analysis. *Bull. Amer. Math. Soc. (N.S.)*, **3** (2):821–843, 1980. ↑

[5] Roger Howe. Quantum mechanics and partial differential equations. *J. Funct. Anal.*, **38** (2):188–254, 1980. ↑

[6] Vladimir V. Kisil. Erlangen programme at large: an Overview. In S.V. Rogosin and A.A. Koroleva (eds.) *Advances in applied analysis*, pages 1–94, Birkhäuser Verlag, Basel, 2012. E-print: arXiv:1106.1686. ↑41

[7] Vladimir V. Kisil. Hypercomplex representations of the Heisenberg group and mechanics. *Internat. J. Theoret. Phys.*, **51** (3):964–984, 2012. E-print: arXiv:1005.5057. Zbl1247.81232. ↑41

[8] Vladimir V. Kisil. Is commutativity of observables the main feature, which separate classical mechanics from quantum?. *Известия Коми научного центра УрО РАН [Izvestiya Komi nauchnogo centra UrO RAN]*, **3** (11):4–9, 2012. E-print: arXiv:1204.1858. ↑41

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