Erlangen program in geometry and analysis

$\mathrm{SL}_2(\mathbb{R})$ case study: Geometry

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Mathematics and the World

Evolution of views

A common habit now is to divide mathematics into pure and applied, the former is considered as a kind of an intellectual game or an exercise in abstract thinking only loosely related to real world.

This attitude is rather recent. Pythagoras said "The whole thing is a number", Euclid wrote his *Elements* about geometry—measurement of land, Newton described heavens with differential equations. Cartesian philosophy considered the Euclidean space-time as the only logical possibility and the true model for the real Newtonian mechanics.

At the first half of XIX century non-Euclidean geometry was discovered, with many other (projective, conformal, differential, ...) to follow. This broke the link between the reality and geometry.

> Why do "non-real" geometries exist?
>
> How many are there of them?

Riemann (1852): *"Local to global" approach:* take an non-degenerate (pseudo-)Riemannian metric g_{ij}. The most of geometry is encoded in the Laplace(–Beltrami) operator, e.g. on the plane:

$$\Delta = \partial_x^2 + \partial_y^2, \qquad \text{and} \qquad \Box = \partial_x^2 - \partial_y^2.$$

With the help of imaginary i and hyperbolic j units the Laplacian and wave operator can be factorised, e.g.:

$$\begin{aligned} \partial_x^2 + \partial_y^2 &= (\partial_x + i\partial_y)(\partial_x - i\partial_y), \qquad i^2 = -1, \\ \partial_x^2 - \partial_y^2 &= (\partial_x + j\partial_y)(\partial_x - j\partial_y), \qquad j^2 = 1. \end{aligned}$$

A connection with analysis: null solutions of Δ are *harmonic* functions and null solutions of D are *holomorphic*.

However, what to do with *parabolic* operators, e.g. $\partial_x^2 + \partial_y$?

Klein's Erlangen program

influenced by S. Lie

Klein (1872): *Geometry is the theory of invariants of a transitive transformation group.*

However the theory of groups and representations was missing at the time, therefore Erlangen program was not in a daily toolkit of a geometer (in contrast to Riemannian geometry).

Family roots of Erlangen program:

- Descartes (1637): geometrical problems reduce to algebraic equations through the coordinate method.
- Galois (1831): solvability of algebraic equations is determined from certain groups.
- S. Lie (1872): solutions of differential equations through their group of continuous symmetries (a marriage of differential geometry with Erlangen approach).

Erlangen Programme at Large

A great manifestation of the Klein's approach is the Special Relativity: the theory of invariants in the Minkowski space-time under the Lorentz transformations.

There is no reason to limit Erlangen programme to geometry or physics. *We shall study invariant properties of functions and operators as well.*

The demonstration can be based on a single (but very important!) example of the group $\mathrm{SL}_2(\mathbb{R})$. Thus we will escape references to general theory of Lie groups and representation, direct calculations will be done instead. There are a lot of open questions and possibilities to generalise this approach we will mention due to course.

Course outline:

- Introduction to $\mathrm{SL}_2(\mathbb{R})$: subgroups and homogeneous spaces.
- Geometry of homogeneous spaces and cycles.
- Linear representations.
- Analytic function theories.
- Functional calculus and quantization.

The group $SL_2(\mathbb{R})$

$SL_2(\mathbb{R})$ is the group of 2×2 matrices with real entries and $\det = 1$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \qquad \text{if} \qquad ad - bc = 1.$$

$SL_2(\mathbb{R})$ is three dimensional: four real parameters minus one constrain. The group law is given by the matrix multiplication, it smoothly depends from parameters. $SL_2(\mathbb{R})$ is non-commutative and non-compact. The group identity is the identity matrix. The inverse defined by:

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad \text{where} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$SL_2(\mathbb{R})$ is the simplest example of semisimple Lie group (the only ideals of its Lie algebra are 0 and the algebra itself). There are other groups connected to $SL_2(\mathbb{R})$, for example $SU(2,2)$, $SO_e(2,1)$, which we only mention briefly here.

One-parameter subgroups
and Lie algebra

Consider a one-parameter continuous subgroup of $SL_2(\mathbb{R})$, that is a collection of elements $g(t) \in SL_2(\mathbb{R})$, $t \in \mathbb{R}$ such that $g(t)g(s) = g(t+s)$.

Here is an example: $g(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad t \in \mathbb{R}.$

Then we can calculate the derivative:

$$\frac{d}{dt} g(t) = \lim_{s \to 0} \frac{g(t+s) - g(t)}{s} = \lim_{s \to 0} \frac{g(s) - e}{s} g(t) = Xg(t),$$

where X turns to be a matrix with zero trace and is called *generator* of the subgroup $g(t)$.

Conversely, any 1-param subgroup $g(t)$ is a solution to the differential equation $g'(t) = Xg(t)$. Thus, we have the representation:

$$g(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n.$$

The collection of all generators is a three-dimensional Lie algebra, closed under commutator $[X, Y] = XY - YX$.

Let X be a set and let be defined an action $G : X \to X$ of G on X. There is an equivalence relation on X, say, $x_1 \sim x_2 \Leftrightarrow \exists g \in G : gx_1 = x_2$, with respect to which X is a disjoint union of distinct *orbits*.

Without loss of a generality, we assume that the operation of G on X is *transitive*, i. e. for every $x \in X$ we have

$$Gx := \bigcup_{g \in G} g(x) = X.$$

If we fix a point $x \in X$ then the set of elements $G_x = \{g \in G \mid g(x) = x\}$ forms the *isotropy (sub)group* of x.

For any group G we could define its action on $X = G$:

- The *conjugation* $g : x \mapsto gxg^{-1}$ (is trivial for a commutative group).
- The *left shift* $\lambda(g) : x \mapsto gx$ and the *right shift* $\rho(g) : x \mapsto xg^{-1}$.

Groups and Homogeneous Spaces

Let G be a group and H be its subgroup. Let us define the space of *cosets* $X = G/H$ by the equivalence relation: $g_1 \sim g_2$ if there exists $h \in H$ such that $g_1 = g_2 h$.

The space $X = G/H$ is a homogeneous space under the left G-action $g : g_1 \mapsto g g_1$. For practical purposes it is more convenient to have a parametrisation of X.

We define a continuous function (section) $s : X \to G$ such that it is a right inverse to the natural projection $p : G \to G/H$, i.e. $p(s(x)) = x$ for all $x \in X$.

Check that, for any $g \in G$, we have $s(p(g)) = gh$ for some $h \in H$ depending from g.

Then, any $g \in G$ has the unique decomposition of the form $g = s(x)h$, where $x = p(g) \in X$ and $h \in H$.

We also define a map $r : g \mapsto h$ associated to p and s by:

$$r(g) := s(x)^{-1} g \text{ where } x = p(g), \text{ thus } g = s(p(g))r(g).$$

$SL_2(\mathbb{R})$ and Its Subgroups

$SL_2(\mathbb{R})$ is the group of 2×2 matrices with real entries and $\det = 1$. A two dimensional subgroup F (F') of lower (upper) triangular matrices:

$$F = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \right\}, \qquad F' = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\}, \qquad a \in \mathbb{R}_+,\ b, c \in \mathbb{R}.$$

F is isomorphic to the group of affine transformations of the real line ($ax + b$ group), isomorphic to the upper half-plane.
There are also three one dimensional continuous subgroups:

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix},\ t \in \mathbb{R} \right\}, \quad (1)$$

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix},\ t \in \mathbb{R} \right\}, \quad (2)$$

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix},\ t \in (-\pi, \pi] \right\}. \quad (3)$$

. . . and Nothing Else

(up to a conjugacy)

Proposition

Any one-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$ *is conjugate to either* A, N *or* K.

Proof.

Any one-parameter subgroup is obtained through the exponentiation

$$e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n \tag{4}$$

of an element X of the Lie algebra $\mathfrak{sl}_2$ of $\mathrm{SL}_2(\mathbb{R})$. Such X is a 2×2 matrix with the zero trace. The behaviour of the Taylor expansion (4) depends from properties of powers X^n. This can be classified by a straightforward calculation. □

Lemma

The square X^2 *of a traceless matrix* $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ *is the identity matrix times* $a^2 + bc = -\det X$*. The factor can be negative, zero or positive, which corresponds to the three different types of the Taylor expansion* (4) *of* $e^{tX} = \sum \frac{t^n}{n!} X^n$.

It is a simple exercise in the Gauss elimination to see that through the matrix similarity we can obtain from X a generator

- of the subgroup K if $(-\det X) < 0$;
- of the subgroup N if $(-\det X) = 0$;
- of the subgroup A if $(-\det X) > 0$.

The determinant is invariant under the similarity, thus these cases are distinct.

$SL_2(\mathbb{R})$ and Homogeneous Spaces

Let G be a group and H be its closed subgroup.
The *homogeneous space* G/H from the equivalence relation: $g' \sim g$ iff $g' = gh$, $h \in H$. The *natural projection* $p : G \to G/H$ puts $g \in G$ into its equivalence class.
A continuous section $s : G/H \to G$ is a right inverse of p, i.e. $p \circ s$ is an identity map on G/H. Then the *left action* of G on itself:

$$\Lambda(g) : g' \mapsto g^{-1} * g', \quad \text{generates} \qquad \begin{array}{ccc} G & \xrightarrow{g*} & G \\ s\uparrow\downarrow p & & s\uparrow\downarrow p \\ G/H & \xrightarrow{g\cdot} & G/H \end{array}$$

If $G = SL_2(\mathbb{R})$ and $H = F$, then $SL_2(\mathbb{R})/F \sim \mathbb{R}$ and $p : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \dfrac{b}{d}$:

$$s : u \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad g : u \mapsto p(g^{-1} * s(u)) = \frac{au + b}{cu + d}, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$SL_2(\mathbb{R})$ and Imaginary Units

Consider $G = SL_2(\mathbb{R})$ and H be any subgroup in the Iwasawa decomp:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \qquad (5)$$

A right inverse s to the natural projection $p : G \to G/H$:

$$s : (u, v) \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \qquad (u, v) \in \mathbb{R}^2, \text{ in the diagram} \qquad \begin{array}{ccc} G & \xrightarrow{g^{-1}*} & G \\ s \uparrow\downarrow p & & s \uparrow\downarrow p \\ G/H & \xrightarrow{g\cdot} & G/H \end{array}$$

defines the G-action $g \cdot x = p(g \cdot s(x))$ on the homogeneous space G/H:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, v) \mapsto \left(\frac{(au + b)(cu + d) - \sigma cav^2}{(cu + d)^2 - \sigma(cv)^2}, \frac{v}{(cu + d)^2 - \sigma(cv)^2} \right).$$

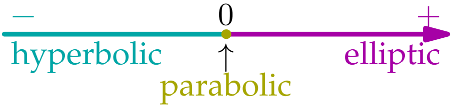
This becomes a Möbius map in (hyper)complex numbers:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw + b}{cw + d}, \qquad w = u + iv, \quad i^2 (:= \sigma) = -1, 0, 1.$$

Elliptic/parabolic/hyperbolic is everywhere

is complex analysis an exception?

Object are subject to the following general classification:



We use representations of $SL_2(\mathbb{R})$ group in function spaces with values in complex, dual and double numbers. Three types of imaginary units:

unit	numbers	case	representation	spectrum	...
$i^2 = -1$,	complex	elliptic	discrete	discrete	...
$\varepsilon^2 = 0$,	dual	parabolic	complementary	residual	...
$j^2 = 1$,	double	hyperbolic	principal	continuous	...

Main distinctions of complex numbers:

- there are no zero divisors (but cannot divide by 0 anyway, so what?)
- are algebraically close (are we using this daily in analysis?)

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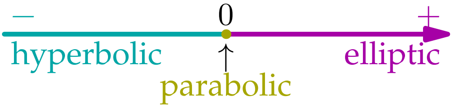

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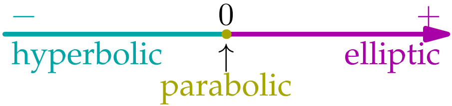

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Möbius Transformations of $\mathbb{R}^2$

For *all* numbers define *Möbius' transformation* of $\mathbb{R}^2 \to \mathbb{R}^2$, (in elliptic and parabolic cases this is even $\mathbb{R}^2_+ \to \mathbb{R}^2_+$!):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ u + iv \ \mapsto \ \frac{a(u+iv)+b}{c(u+iv)+d}. \tag{6}$$

Product $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ gives *Iwasawa* $SL_2(\mathbb{R}) = ANK$. In all $\mathbb{A}$ subgroups A and N acts uniformly:

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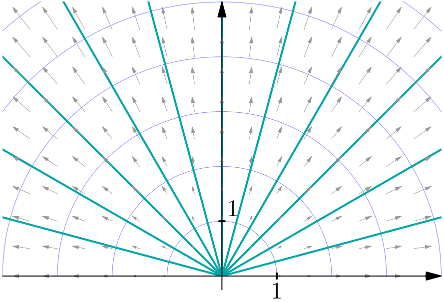

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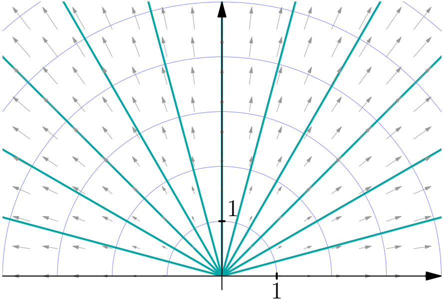
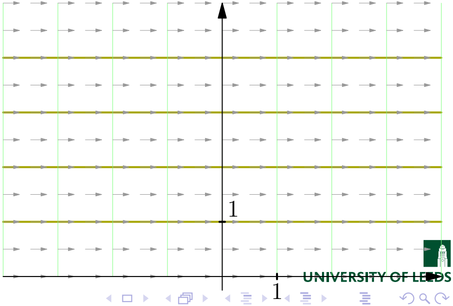
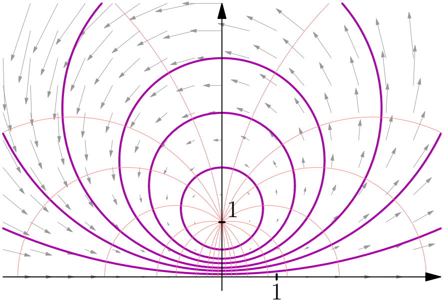


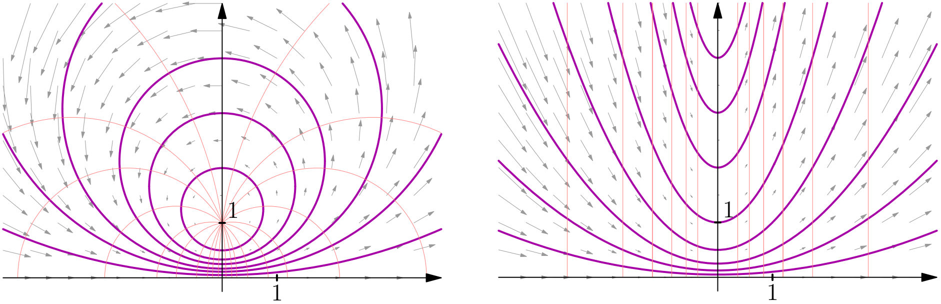




Vector fields are:

$$\begin{aligned} dK_e(u,v) &= (1+u^2-v^2, & 2uv) \\ dK_p(u,v) &= (1+u^2, & 2uv) \\ dK_h(u,v) &= (1+u^2+v^2, & 2uv) \\ dK_\sigma(u,v) &= (1+u^2+\sigma v^2, & 2uv) \end{aligned}$$

Figure: Depending from $i^2=\sigma$ the orbits of subgroup K are circles, parabolas and hyperbolas passing $(0,t)$ with the equation $(u^2-\sigma v^2)+v(\sigma t-t^{-1})+1=0$. This leads to elliptic, parabolic and hyperbolic analytic functions.



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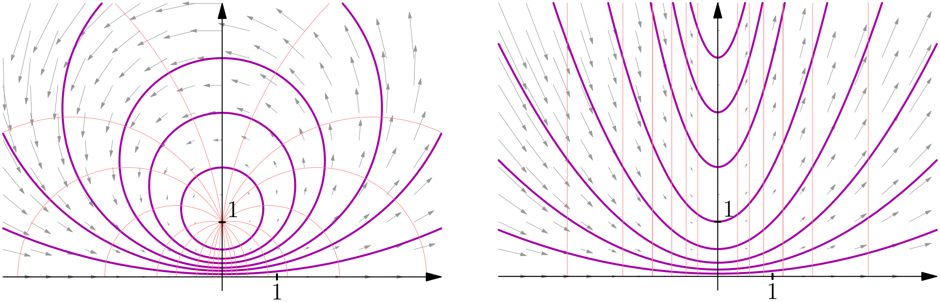

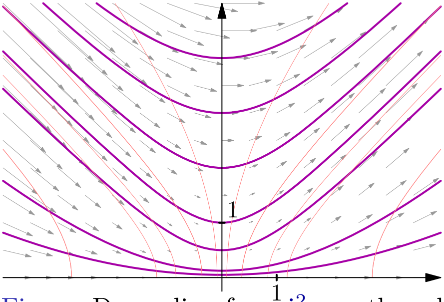

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Unification in higher dimensions

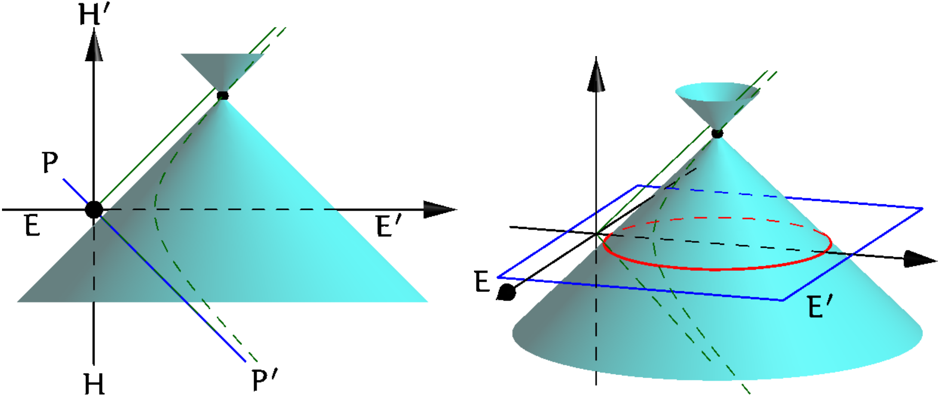


Figure: K-orbits as conic sections: elliptic case

Parabolic and hyperbolic sections

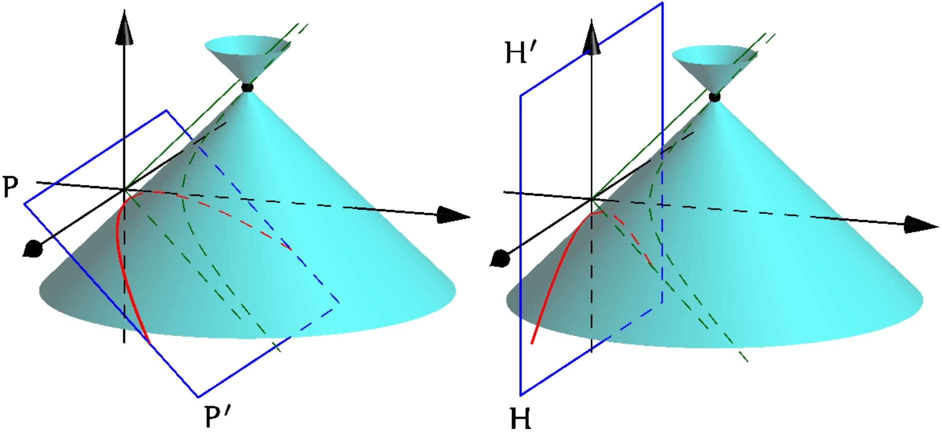


Figure: K-orbits as conic sections: parabolic and hyperbolic cases

The cone rotated as the whole

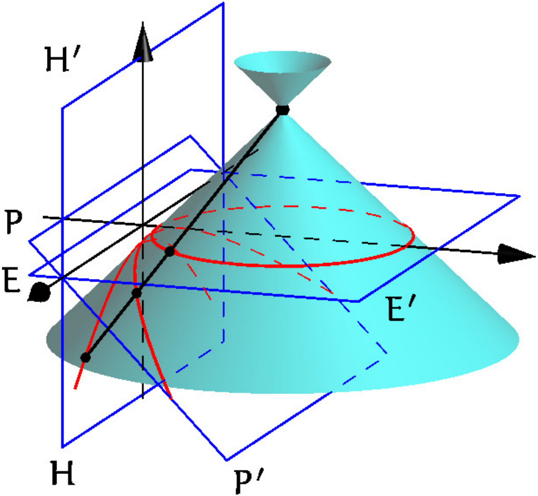
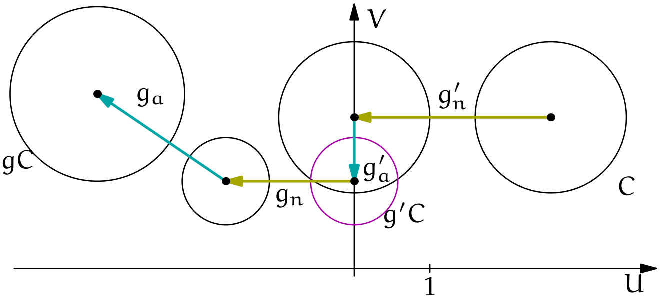


Figure: K-orbits as conic sections: a cone generator passing three orbits at the same values of ϕ.

Lemma

Möbius transformations preserve the cycles (circles, parabolas, hyperbolas in respective cases) in the upper half plane.

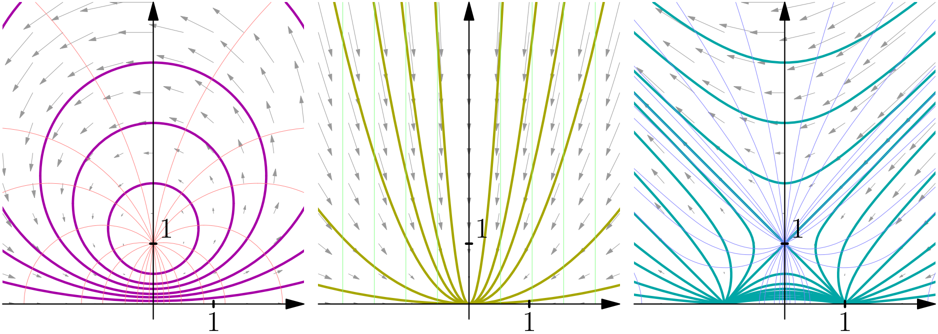


Proof.

Subgroups A and N obviously preserve all cycles in $\mathcal{C}\ell(a)$. And K preserves cycles which are its own orbits.

Thus we decompose an arbitrary Möbius transformation g into a product $g = g_a g_n g_k g'_a g'_n$, where the existence of $g_a g_n g_k = g g_n'^{-1} g_a'^{-1}$, is provided by the Iwasawa decomposition.

Fix subgroups of i, ε and j



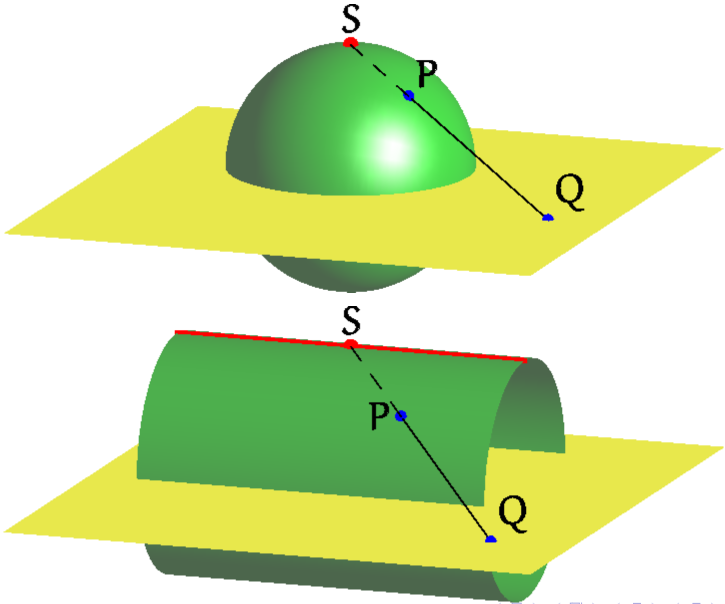
$$K = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad N' = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \qquad A' = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

Fix subgroups of $(0, 1)$ are $S(t) = \exp \begin{pmatrix} 0 & \sigma t \\ t & 0 \end{pmatrix}$, that is

$K = \exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$—elliptic, $N' = \exp \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$—parabolic and

$A' = \exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$—hyperbolic cases.

Compactification of $\mathbb{R}^e$ and $\mathbb{R}^p$



Compactification of $\mathbb{R}^{h}$

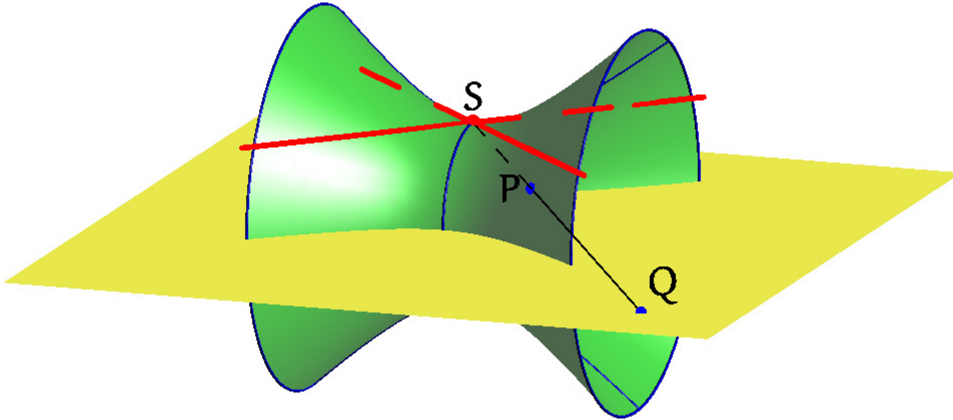


Figure: Hyperbolic counterpart of the Riemann sphere (incomplete so far!) Ideal elements for the *light cone* at infinity.

In all EPH cases ideal points comprise the corresponding zero-radius cycle at infinity.

Schwerdtfeger–Fillmore–Springer–Cnops

Construction

Cycles may be put to correspondence with certain matrices:

$$k(u^2 - \breve{\imath}^2 v^2) - 2\langle (l, n), (u, v) \rangle + m = 0 \quad \longleftrightarrow \quad \begin{pmatrix} l + \breve{\imath} s n & -m \\ k & -l + \breve{\imath} s n \end{pmatrix}$$

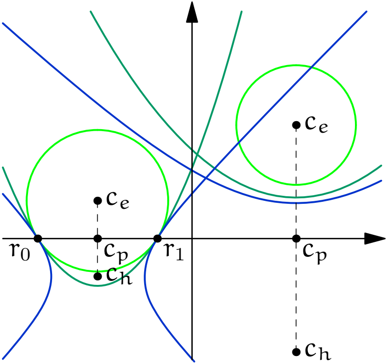


Figure: Different EPH implementations of the same cycles

A cycle is defined by four numbers $(k, l, n, m) \in \mathbb{R}^4$ up to a scalar factor, i.e. this is a vector in a projective space.

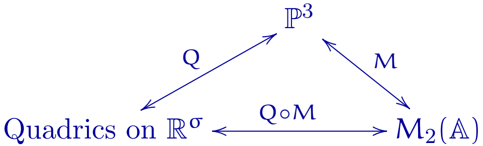
For any such vector there three possibility to draw a cycle as circle, parabola or hyperbola.

Its centre is at the point $\left(\frac{l}{k}, -\breve{\sigma}\frac{n}{k}\right)$

Cycles: Three Faces

and two normalisations

We have the following maps between different faces of cycles:



We may try to make the correspondence 1-1, that is remove projectivity by normalisation. However this cannot be done globally.

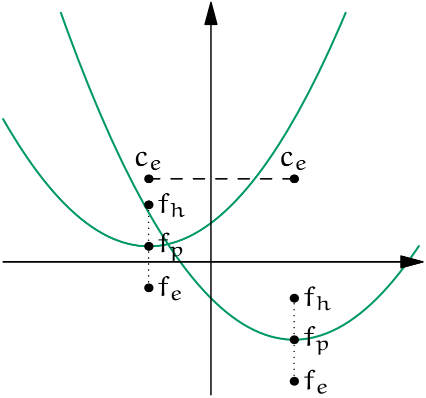
Definition

If $k \neq 0$ we may use the *leading normalisation*: the quadruple to $(1, \frac{l}{k}, \frac{n}{k}, \frac{m}{k})$ with highlighted *centre of a cycle*. Moreover in this case $\det C^s_{\breve{\sigma}}$ is equal to the square of *cycle radius*.

Kirillov's normalisation is give by $\det C^s_{\breve{\sigma}} = 1$. It highlights curvature and gives a nice condition for touching circles.

Foci of a cycle

Then the classical invariants of a matrix (the *trace* and *determinant*) will represent some geometric invariants of cycles.



The determinant

$$D_{\breve{\sigma}} = km - l^2 + \breve{\sigma} n^2$$

of matrix $\begin{pmatrix} l + \breve{\imath} sn & -m \\ k & -l + \breve{\imath} sn \end{pmatrix}$ plays the important role.

For example, foci of a cycle is defined as $\left(\frac{l}{k}, \frac{D_{\breve{\sigma}}}{k^2}\right)$. Foci are defined by this expression also for circles and hyperbolas but are different from the classic foci.

Figure: Foci of two parabolas with the same focal length.

Zero-radius cycles

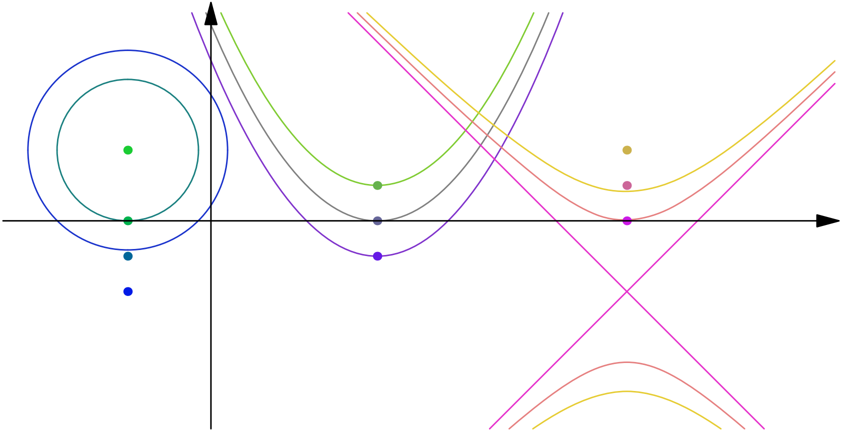


Figure: Different σ-implementations of the same $\breve{\sigma}$-zero-radius cycles, i.e. $\det C^s_{\breve{\sigma}} = 0$. Any implementation of p-zero-radius cycle touches the real line. The corresponding focus belongs to the real line as well. In the case $\sigma\breve{\sigma} = 1$ zero-radius cycles are either point (elliptic case) or the light cone at the point.

Linearisation of Möbius maps

Theorem

Let a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ *defines a Möbius transformation*

$$M : (u + iv) \to \frac{a(u + iv) + b}{c(u + iv) + d}.$$

Then any cycle C *defined* $k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0$ *is transformed to a cycle* C_2 *associated to the matrix* MCM^{-1}, *where* $C = \begin{pmatrix} l + \breve{\imath}sn & -m \\ k & -l + \breve{\imath}sn \end{pmatrix}$.

Proof.

A proof can be done through algebraic manipulation by a CAS. Is this the ultimate goal of Descartes' coordinate method?

An alternative smart reasoning is based on the orthogonality of cycles which is described below.

Inner product and Orthogonality

The orthogonality is defined by the condition:

$$\left\langle C^{s}_{\breve{\sigma}}, \tilde{C}^{s}_{\breve{\sigma}} \right\rangle = 0, \qquad \text{where } \left\langle C^{s}_{\breve{\sigma}}, \tilde{C}^{s}_{\breve{\sigma}} \right\rangle = \Re\,\mathrm{tr}(C^{s}_{\breve{\sigma}} \tilde{C}^{s}_{\breve{\sigma}}).$$

which is **Möbius invariant**, since it is invariant under matrix conjugation.

This is exactly the definition used in the GNS construction in C^*-algebras. The explicit expression for the inner product is:

$$\left\langle C^{s}_{\breve{\sigma}}, \tilde{C}^{s}_{\breve{\sigma}} \right\rangle = 2\breve{\sigma}\tilde{n}n - 2\tilde{l}l + \tilde{k}m + \tilde{m}k.$$

For matrices representing cycles we obtain the second classical invariant (determinant) under similarities from the first (trace) as follows:

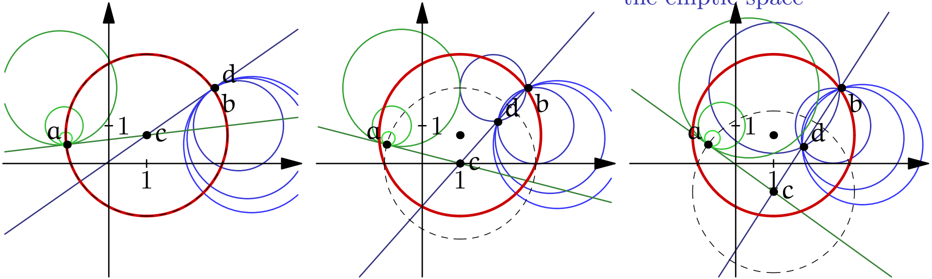
$$\langle C^{s}_{\breve{\sigma}}, C^{s}_{\breve{\sigma}} \rangle = -2 \det C^{s}_{\breve{\sigma}}.$$

To describe ghost cycle we need the *Heaviside function* $\chi(\sigma)$:

$$\chi(t) = \begin{cases} 1, & t \geqslant 0; \\ -1, & t < 0. \end{cases}$$

Orthogonality of cycles

the elliptic space



$\sigma = -1, \breve{\sigma} = -1$ $\sigma = -1, \breve{\sigma} = 0$ $\sigma = -1, \breve{\sigma} = 1$

Theorem

A cycle is $\breve{\sigma}$*-orthogonal to cycle* $C^s_{\breve{\sigma}}$ *if it is orthogonal in the usual sense to the* σ*-realisation of "ghost" cycle* $\hat{C}^s_{\breve{\sigma}}$*, which is defined by the following two conditions:*

1. $\chi(\sigma)$*-centre of* $\hat{C}^s_{\breve{\sigma}}$ *coincides with* $\breve{\sigma}$*-centre of* $C^s_{\breve{\sigma}}$*.*
2. *Cycles* $\hat{C}^s_{\breve{\sigma}}$ *and* $C^s_{\breve{\sigma}}$ *have the same roots, moreover* $\det \hat{C}^1_{\sigma} = \det C^{\chi(\breve{\sigma})}_{\sigma}$.

Orthogonality of cycles in the hyperbolic space

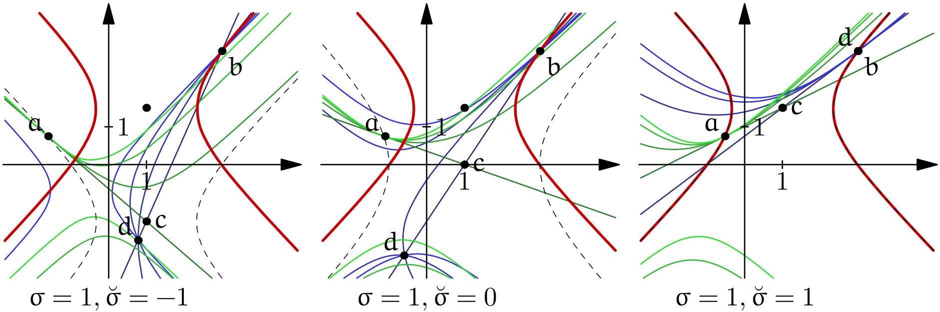


Figure: Orthogonality of the first kind in hyperbolic point space and EPH type of the cycle space

The orthogonality is defined by the same condition $\left\langle C^{s}_{\breve{\sigma}}, \tilde{C}^{s}_{\breve{\sigma}} \right\rangle = 0$.
For the matching case $\sigma = \breve{\sigma}$ it is locally $uu' - \sigma vv' = uu' - vv' = 0$.

Parabolic orthogonality of cycles

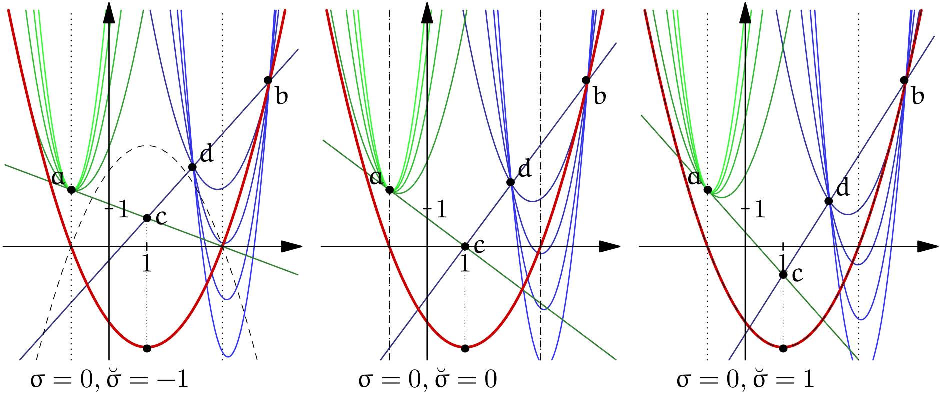
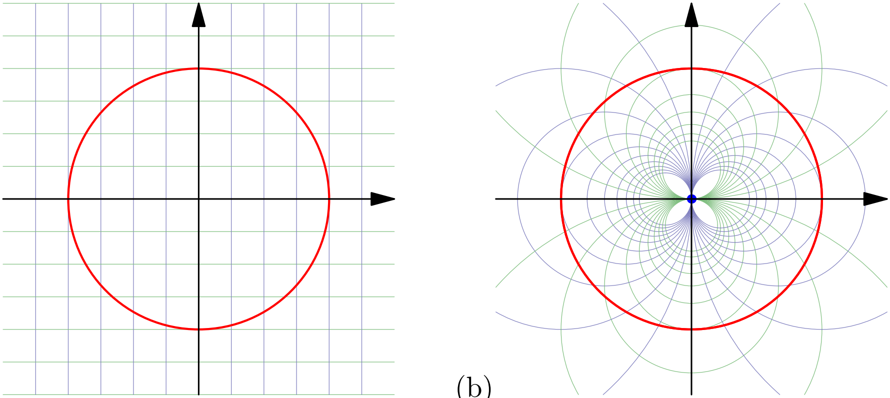


Figure: Orthogonality of the first kind in parabolic point space and EPH type of the cycle space. Note intersection of lines in the centre of the red parabola.

This orthogonality is defined by the same condition $\left\langle C^s_{\breve{\sigma}}, \tilde{C}^s_{\breve{\sigma}} \right\rangle = 0$. It is **not** reduced locally to $uu' = uu' - \sigma vv' = 0$.

Elliptic Inversion in a cycle



(a) (b)

Figure: The initial rectangular grid (a) is inverted elliptically in the unit circle (shown in red) on (b). The blue cycle (collapsed to a point at the origin on (b)) represent the image of the cycle at infinity under inversion.

Parabolic and Hyperbolic Inversions

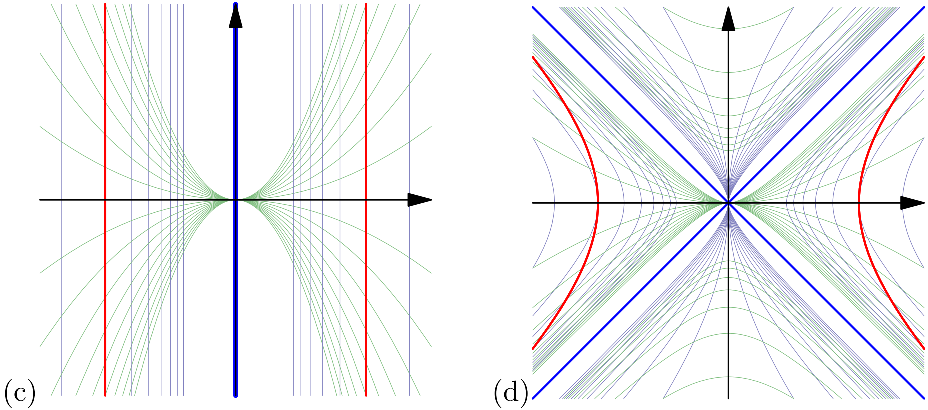


Figure: The initial rectangular grid (a) is inverted in the unit circle parabolically on (c) and hyperbolically on (d). The blue cycle represent the image of the cycle at infinity under inversion.

Higher order invariants

For any polynomial $p(x_1, x_2, \ldots, x_n)$ of several non-commuting variables one may define an invariant joint disposition of n cycles ${}^{j}C^{s}_{\breve{\sigma}}$ by the condition:

$$\operatorname{tr} p({}^{1}C^{s}_{\breve{\sigma}}, {}^{2}C^{s}_{\breve{\sigma}}, \ldots, {}^{n}C^{s}_{\breve{\sigma}}) = 0, \qquad (7)$$

there all cycles are used in Kirillov's normalisation.
It is clear, that the condition (7) is $SL_2(\mathbb{R})$-invariant, i.e preserved by the transformations $C^{s}_{\breve{\sigma}} \mapsto MC^{s}_{\breve{\sigma}}M^{-1}$, $M \in SL_2(\mathbb{R})$. Moreover, in this matrix similarity we can replace element $M \in SL_2(\mathbb{R})$ by an arbitrary matrix corresponding to another cycle:

Lemma

The product $C^{s}_{\breve{\sigma}}\tilde{C}^{s}_{\breve{\sigma}}C^{s}_{\breve{\sigma}}$ is again the FSCc matrix of a cycle. This cycle may be considered as the reflection of $\tilde{C}^{s}_{\breve{\sigma}}$ in $C^{s}_{\breve{\sigma}}$.

Remark

We can reduce the order of invariant (7) replacing some of cycles ${}^{j}C^{s}_{\breve{\sigma}}$ by the real line, which is $SL_2(\mathbb{R})$-invariant as well.

We illustrate the above construction by the following invariant of third order which is obtained from some fourth order invariant by the reduction with real line.

Definition

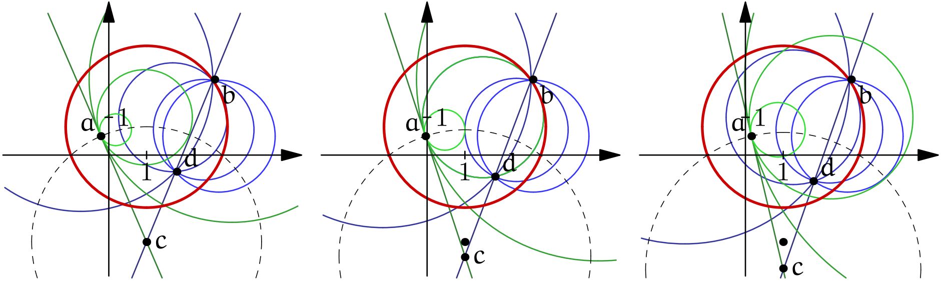
A cycle $C^s_{\breve{\sigma}}$ is f-orthogonal *to* a cycle $\tilde{C}^s_{\breve{\sigma}}$ if the reflection of $\tilde{C}^s_{\breve{\sigma}}$ in $C^s_{\breve{\sigma}}$ is orthogonal (in the previous sense) to the real line.
Analytically this is defined by:

$$\mathrm{tr}(C^s_{\breve{\sigma}}\tilde{C}^s_{\breve{\sigma}}C^s_{\breve{\sigma}}R^s_{\breve{\sigma}}) = 0. \quad (8)$$

Remark

It is important to note that this is a non-commutative relation: there is C_1 which is f-orthogonal to C_2 but C_2 is not f-orthogonal to C_1. This will be later a source of a non-commutative distance: $|AB| \neq |BA|$. As we will see, this fancy distance still possesses a conformal property!

Focal Orthogonality (elliptic)



$\sigma = -1, \breve{\sigma} = -1$ $\sigma = -1, \breve{\sigma} = 0$ $\sigma = -1, \breve{\sigma} = 1$

Theorem

A cycle is s-orthogonal to a cycle $C^s_{\breve{\sigma}}$ *if its orthogonal in the traditional sense to its s-ghost cycle* $\tilde{C}^{\breve{\sigma}}_{\breve{\sigma}} = C^{\chi(\sigma)}_{\breve{\sigma}} \mathbb{R}^{\breve{\sigma}}_{\breve{\sigma}} C^{\chi(\sigma)}_{\breve{\sigma}}$, *which is the reflection of the real line in* $C^{\chi(\sigma)}_{\breve{\sigma}}$ *and* χ *is the Heaviside function. Moreover*

1. $\chi(\sigma)$*-Centre of* $\tilde{C}^{\breve{\sigma}}_{\breve{\sigma}}$ *coincides with the* $\breve{\sigma}$*-focus of* $C^s_{\breve{\sigma}}$, *consequently all lines s-orthogonal to* $C^s_{\breve{\sigma}}$ *are passing the respective focus.*
2. *Cycles* $C^s_{\breve{\sigma}}$ *and* $\tilde{C}^{\breve{\sigma}}_{\breve{\sigma}}$ *have the same roots.*

Focal Orthogonality

hyperbolic space

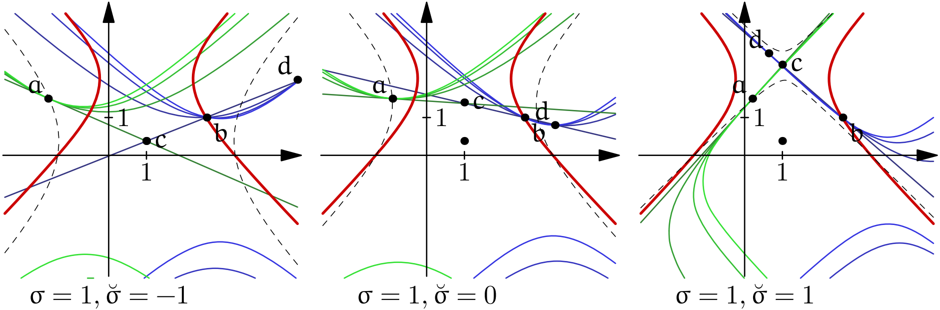


Figure: Focal orthogonality in the hyperbolic space: two pencils (green and blue) of hyperbolas orthogonal to the red one. The ghost hyperbola is dashed.

Focal Orthogonality

parabolic space

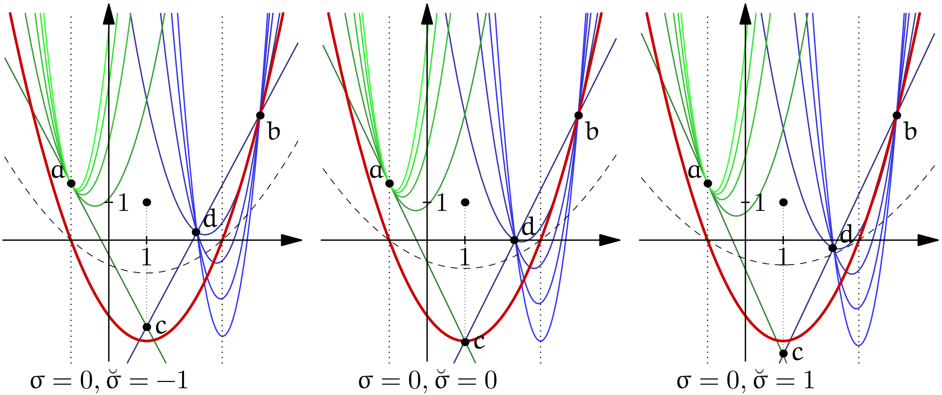


Figure: Focal orthogonality in the parabolic space: two pencils (green and blue) of parabolas orthogonal to the red one. The ghost parabola is dashed, its centre is evidently coincides with some focus of the red parabola.

Diameters and distances

Definition

The diameter of the cycle is determinant of its matrix.

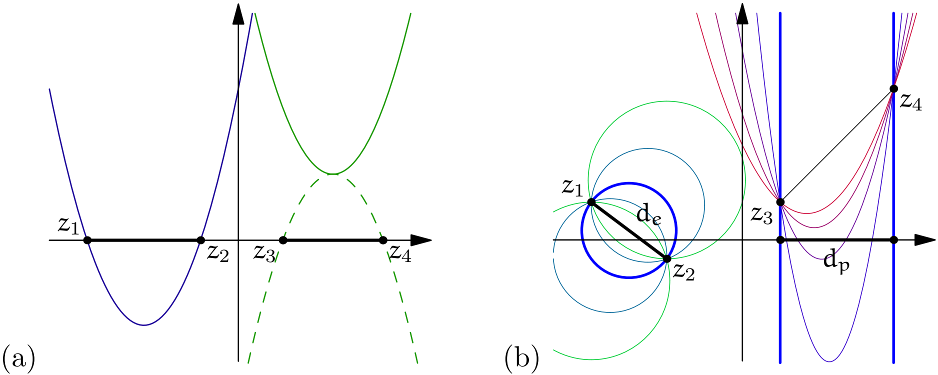
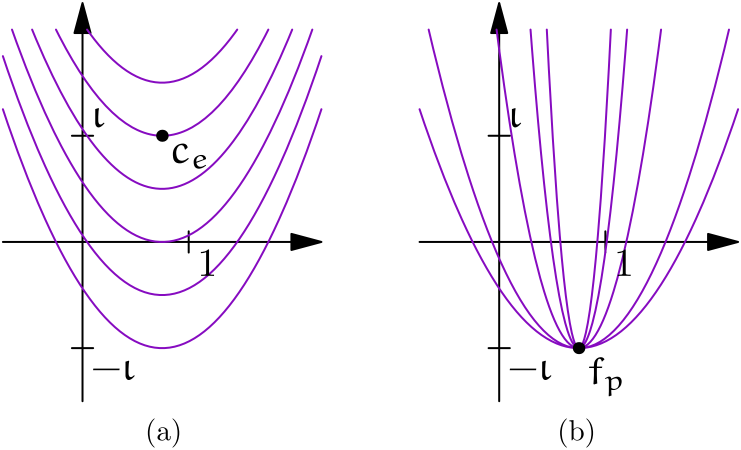


Figure: (a) Square of the parabolic diameter is square of the distance between roots if they are real (z_1 and z_2), otherwise minus square of the distance between the adjoint roots (z_3 and z_4).
(b) Distance as extremum in elliptic (z_1 and z_2) and parabolic (z_3 and z_4) cases.

Lengths from Centres and Foci

Definition

A length of a *directed* interval AB is the half-diameter of a cycle with eph-centre or eph-focus at A passing B. There are $3 \times 3 \times 3 \times 2 = 54$.



(a) Concentric parabolas "shrinking around" c_e .

(b) Co-focal parabolas "shrinking around" f_p.

Conformality of Distances/Lengths

Definition

We say that a distance or a length d is $SL_2(\mathbb{R})$-*conformal* if for fixed y, $y' \in \mathbb{R}^\sigma$ the limit:

$$\lim_{t\to 0} \frac{d(g\cdot y, g\cdot(y+ty'))}{d(y, y+ty')}, \qquad \text{where } g \in SL_2(\mathbb{R}), \tag{9}$$

exists and its value is independent from y'.

The following proposition shows that $SL_2(\mathbb{R})$-conformality is not rare.

Proposition

1. *The distance are conformal if and only if the type of point and cycle spaces are the same.*
2. *The lengths from centres and foci are conformal for any combination of point space, cycle space and centre/focus.*

Isometric Action

Q. Is their a distance in the upper half-plane such that Möbius transformations are isometries?
A. Yes: $d^2(w_1, w_2) = \mathsf{F}\left(\frac{(w_1 - w_2)(\bar{w}_1 - \bar{w}_2)}{\Im w_1 \cdot \Im w_2}\right)$

Definition (L.M. Blumenthal)

Line is a curve such that distance is additive along it.

Theorem

The lines in parabolic geometry with the distance function $\sin^{-1}_{\breve{\sigma}} \frac{|z-w|_p}{2\sqrt{\Im[z]\Im[w]}}$ *are parabolas of the form* $(\breve{\sigma} + 4t^2)u^2 - 8tu - 4v + 4 = 0$, *where:*

$$\sin^{-1}_{\breve{\sigma}} x = \begin{cases} \sinh^{-1} x, & \text{if } \breve{\sigma} = -1.; \\ 2x, & \text{if } \breve{\sigma} = 0; \\ \sin^{-1} x, & \text{if } \breve{\sigma} = 1. \end{cases}$$

Parabolic Lines

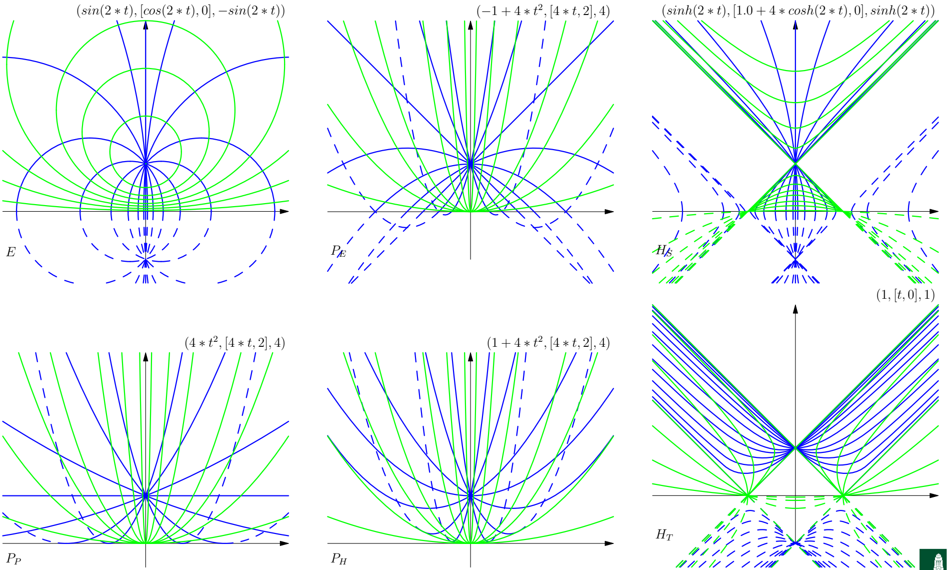


Figure: Geodesics (blue) and equidistant orbits (green) in EPH geometries.

0 and ε cycles, perpendicularity

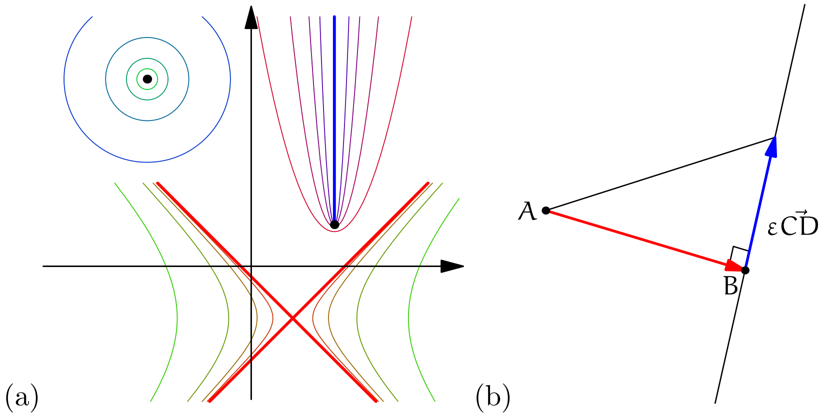


Figure: (a) Zero-radius cycles in elliptic (black point) and hyperbolic (the red light cone). Infinitesimal radius parabolic cycle is the blue vertical ray starting at the focus. For infinitesimal cycles *focal* orthogonality (Fig. 56) is replacement of usual one for zero-radius cycles.
(b) Perpendicularity as the shortest distance. *Focal* orthogonality coincides with it!

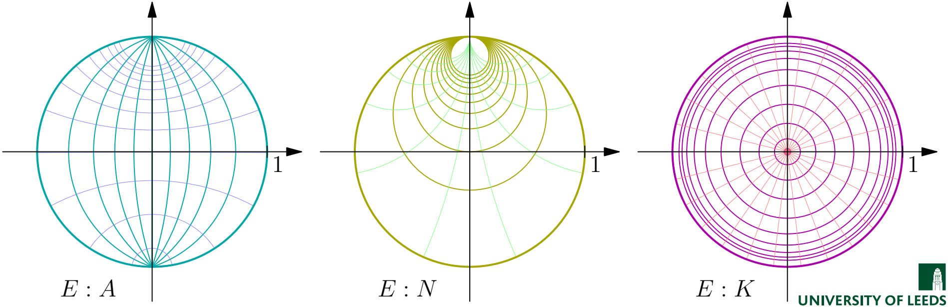
Cayley Transform and Unit "Circles"

The colour code of ANK match to the model, where subgroup is diagonalised.

In elliptic case the standard Cayley transform diagonalises K:

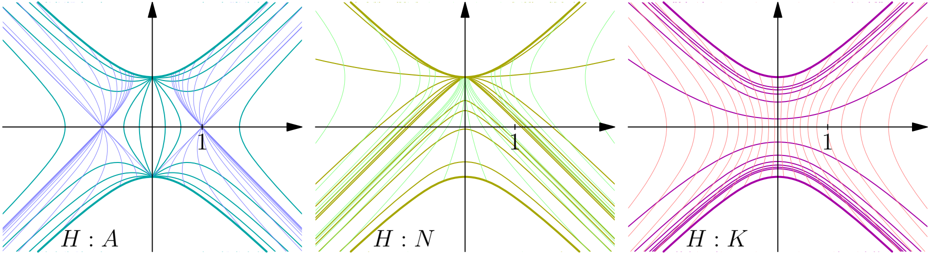
$$\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = \frac{1}{\sqrt{1-|u|^2}} \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} \begin{pmatrix} 1 & \bar{u} \\ u & 1 \end{pmatrix}, \text{ with } \begin{matrix} \omega = \arg \alpha, \\ u = \beta \bar{\alpha}^{-1}, \end{matrix} \text{ by } \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and $|u| < 1$ follows from $|\alpha|^2 - |\beta|^2 = 1$. $i^2 = -1$. Cf. Figs. 6 and 1.



In hyperbolic case we analogously diagonalise A:

$$\begin{pmatrix} \mathsf{a} & \mathsf{b} \\ -\mathsf{b} & \mathsf{a} \end{pmatrix} = |\mathsf{a}| \begin{pmatrix} \frac{\mathsf{a}}{|\mathsf{a}|} & 0 \\ 0 & \frac{\mathsf{a}}{|\mathsf{a}|} \end{pmatrix} \begin{pmatrix} 1 & \mathsf{a}^{-1}\mathsf{b} \\ -\mathsf{a}^{-1}\mathsf{b} & 1 \end{pmatrix} \qquad \text{by} \begin{pmatrix} 1 & \mathsf{i} \\ -\mathsf{i} & 1 \end{pmatrix}$$



However we could not deduce $|\mathsf{a}^{-1}\mathsf{b}| < 1$ now! (Figure 1 cheats)

Geometry: $\mathbb{R}^2$ is not split by the unit circle into "interior" and "exterior",

Analysis: Hardy space is not a proper subset of L_2;

Physics: Past and future could be reversed continuously.

Future-to-past continuous transform

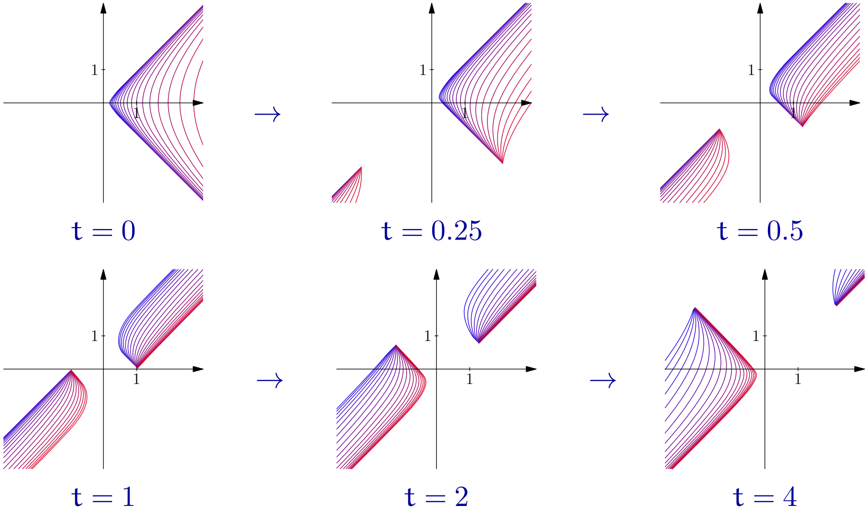
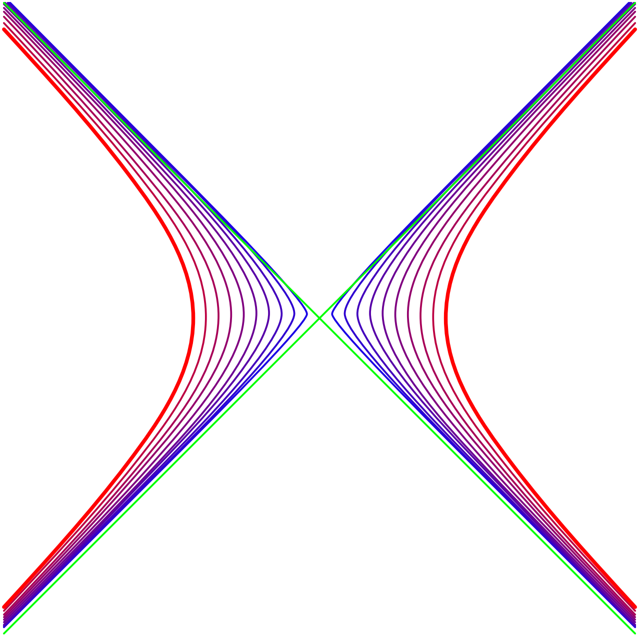


Figure: Eight frames from a continuous family of transformation which inverts the hyperbolic unit disk without a self-intersection in the finite part of the plane. It also revert the direction of the time-arrow.



Double cover of a hyperbolic plane

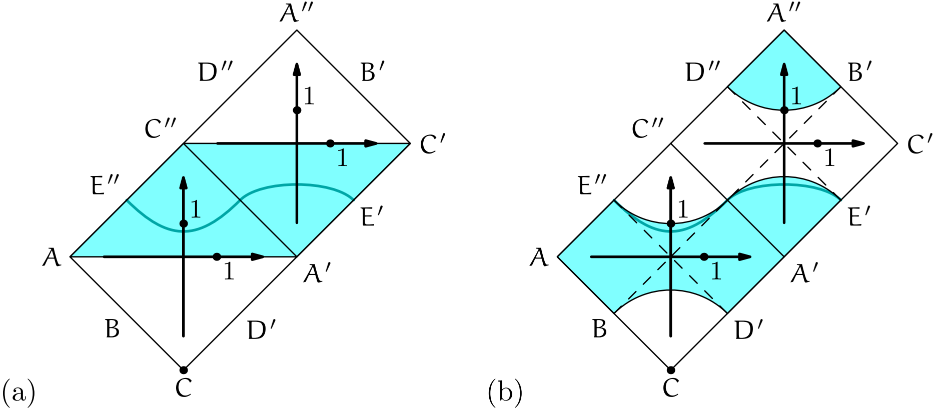


Figure: Hyperbolic objects in the double cover of $\mathbb{R}^{h}$:
(a) the “upper” half-plane;
(b) the interior part of the unit circle.

Compactification of $\mathbb{R}^h$

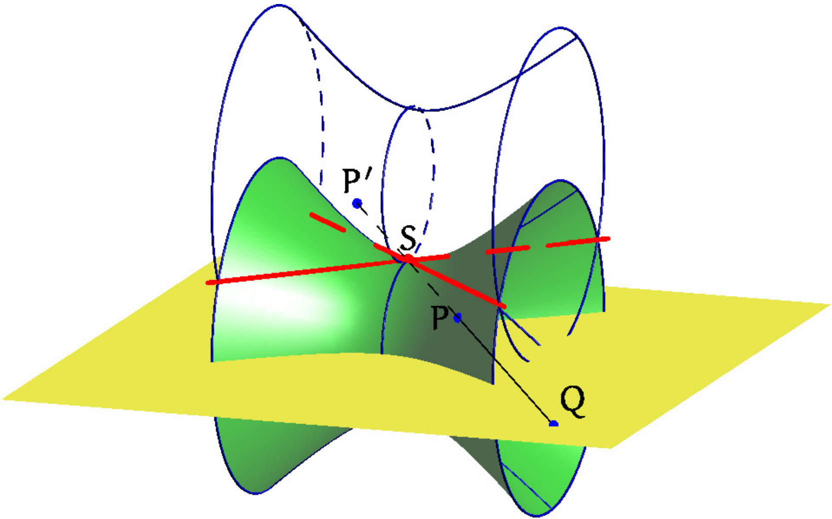
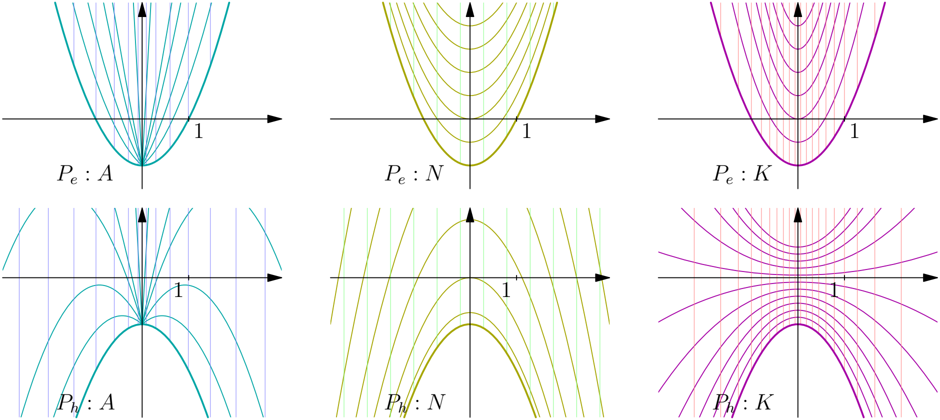


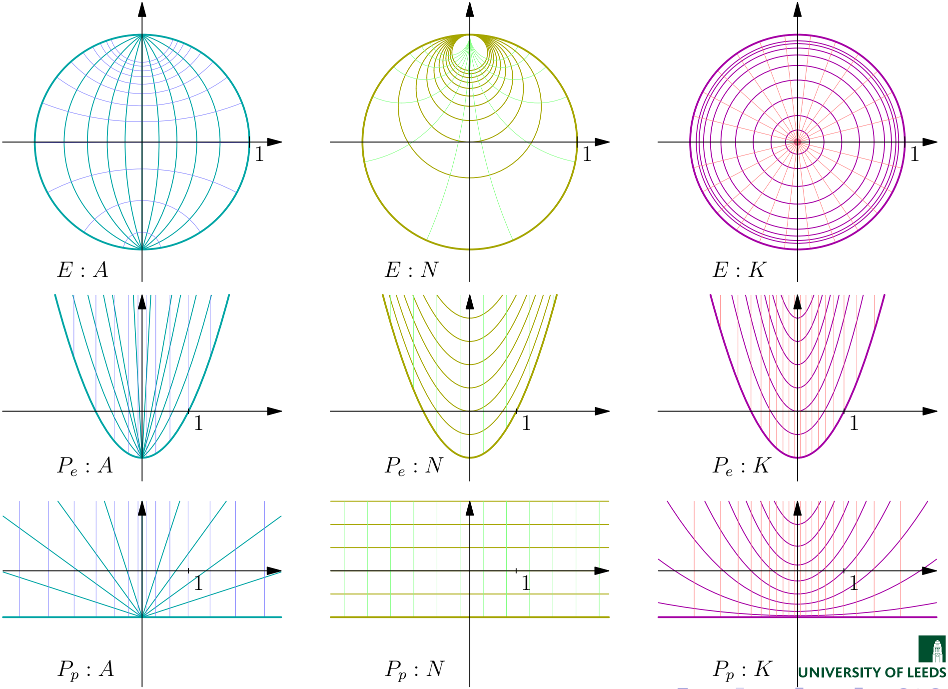
Figure: Hyperbolic counterpart of the Riemann sphere, complete this time!

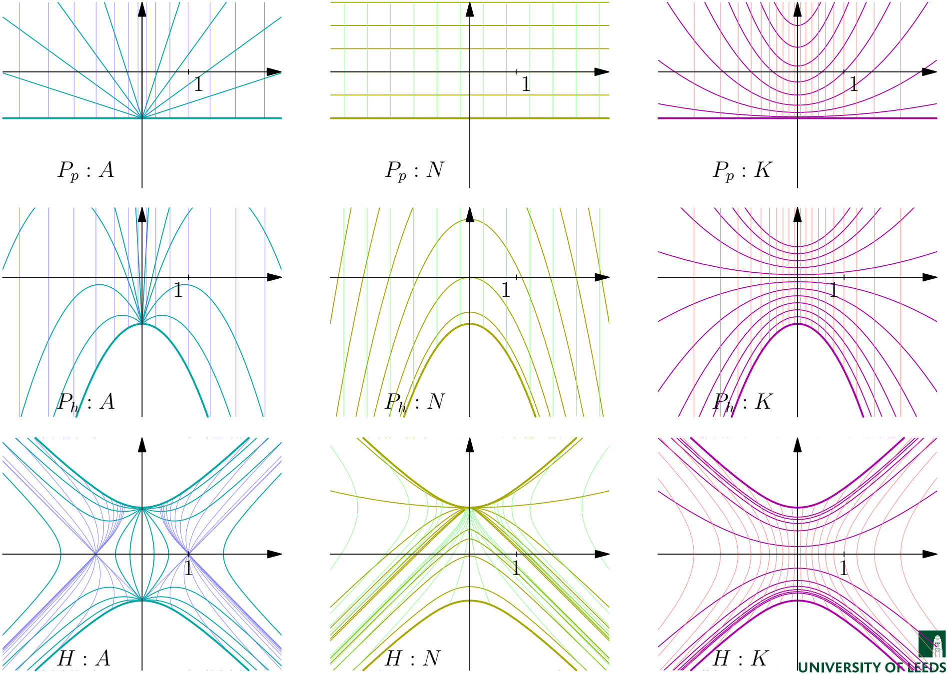
Parabolic unit circles

The initial action of the subgroup N is already “diagonalised” as much as possible: thus the upper half plane is already a parabolic “unit circle”, however we have also elliptic and hyperbolic versions:



Reason: only for the degenerate $\mathrm{i}^2 = 0$ both matrices $\begin{pmatrix} 1 & -\mathrm{i} \\ \pm\mathrm{i} & 1 \end{pmatrix}$ are not!


$E : A$
$E : N$
$E : K$
$P_e : A$
$P_e : N$
$P_e : K$
$P_p : A$
$P_p : N$
$P_p : K$
1


$P_p : A$
$P_p : N$
$P_p : K$
$P_h : A$
$P_h : N$
$P_h : K$
$H : A$
$H : N$
$H : K$
1

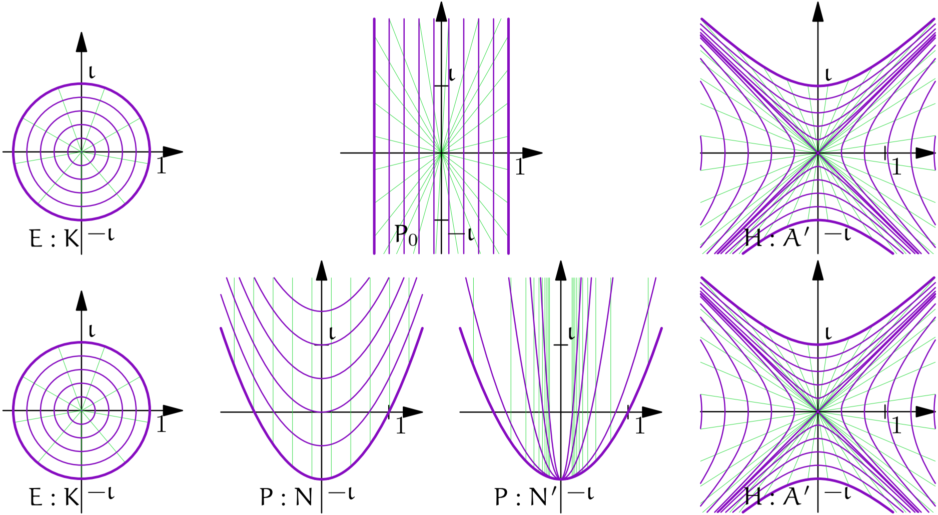


Figure: Rotation as multiplication (the first row) and Möbius transformations (the second row). Orbits are level lines for the respective norm. Straight lines join points with the same value of "angle" (argument). Note that orbits of the subgroup N are *concentric* parabolas and orbits of N′—are co-focal ones. They are Cayley transforms of the fix subgroups of the point (0, 1) from Fig. 37

Induced Representations

Let G be a group, H its closed subgroup, χ be a linear representation of H in a space V. The set of V-valued functions with the property

$$F(gh) = \chi(h)F(g),$$

is invariant under left shifts.
The restriction of the left regular representation to this space is called an *induced representation*.
Equivalently we consider the *lifting* of $f(x)$, $x \in X = G/H$ to $F(g)$:

$$F(g) = \chi(h)f(p(g)), \qquad p : G \to X, \quad g = s(x)h, \quad p(s(x)) = x.$$

This is a 1-1 map which transform the left regular representation on G to the following action:

$$[\rho'(g)f](x) = \chi(h)f(g \cdot x), \qquad \text{where} \quad gs(x) = s(g \cdot x)h.$$

In the case of $\mathrm{SL}_2(\mathbb{R})$ we have three different types of actions.