Erlangen program in geometry and analysis

$\mathrm{SL}_2(\mathbb{R})$ case study: Analytic Functions

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Why $i^2 = -1$ not 1 or 0? Let us try!
We wish to have "analytic function theory"!!! Naturally:

$$f(x+iy) = \sum_{k=0}^{\infty} a_k (x+iy)^k \quad \text{is analytic extension of} \quad f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

For $j^2 = 1$ we have $e_\pm = \frac{1}{2}(1 \pm j)$ such that $e_+ e_- = 0$, $e_\pm^2 = e_\pm$, then

$$(x+jy)^k = ((x+y)e_+ + (x-y)e_-)^k = (x+y)^k e_+ + (x-y)^k e_-.$$

Thus any "hyperbolic" analytic function has the form

$$f(x+jy) = f(x+y)e_+ + f(x-y)e_-. \quad \text{(wave eq. solution at least...)}$$

Cauchy "integral" formulae—dependence from one or two points on the real line.

Parabolic Analytic Theory

Even simpler :-(or trivial, in fact

For $\varepsilon^2 = 0$, we have from the binomial formula

$$(x + \varepsilon y)^k = x^k + \varepsilon k x^{k-1} y.$$

Thus, from the Taylor expansion, a "parabolic" analytic function is

$$f(x + \varepsilon y) = f(x) + \varepsilon f'(x) y \quad \text{(is not even heat eq. solution...)}$$

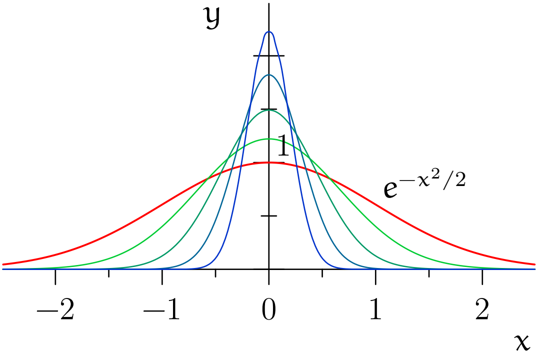
The Cauchy "integral" formulae—dependence from one point on real line. Yet, this is an incarnation of *non-standard analysis*, i.e. ε^2 is negligible at ε-scale.

There are successful constructions of hyperbolic and parabolic analytic theories through

- extension of the elliptic case (Sprößig, Gürlebeck, . . .);
- more advanced Clifford algebra (Cerejeiras, Kähler, Sommen, . . .).

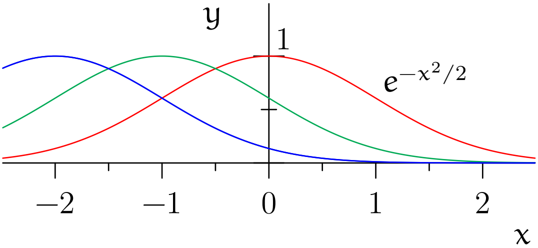
However, nothing interesting seems to be possible from dual and double numbers.

Wavelets from the $ax + b$ group



Scaling and shift (the affine group) create a family of wavelets, which are used for wavelet transform:

$$\begin{aligned} \hat{f}(a,b) &= \left\langle f, \rho_{(a,b)}\phi_0 \right\rangle \\ &= \left\langle f, \phi_{(a,b)} \right\rangle \end{aligned}$$



The Gaussian $\phi(x) = e^{-x^2/2}$ as a mother wavelet produces an approximation of δ-function.

The mother wavelet $\phi(x) = \frac{1}{x+i}$ generates the Cauchy integral formula for the upper half-plane.

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Wavelets and Groups

Let G be a group and ρ be its unitary irreducible representation in a Hilbert space H. For a fixed *mother wavelet* $w_0 \in H$ define *wavelet transform* from H to $C_b(G)$:

$$[\mathcal{W}f](g) = \langle f, \rho_g w_0 \rangle = \left\langle \rho_{g^{-1}} f, w_0 \right\rangle, \quad g \in G.$$

Let Λ be the *left regular representation* on G:

$$\Lambda(g) : f(h) \to f(g^{-1}h).$$

The following properties of the wavelet transform are of interest:

Proposition

1. $\mathcal{W}$ *intertwines* ρ *and* Λ*:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}.$$

2. *The image of* $\mathcal{W}$ *is invariant under left shifts.*
3. *The image of* $\mathcal{W}$ *is spanned by translations of* $\mathcal{W}w_0$*—the image of the mother wavelet .*

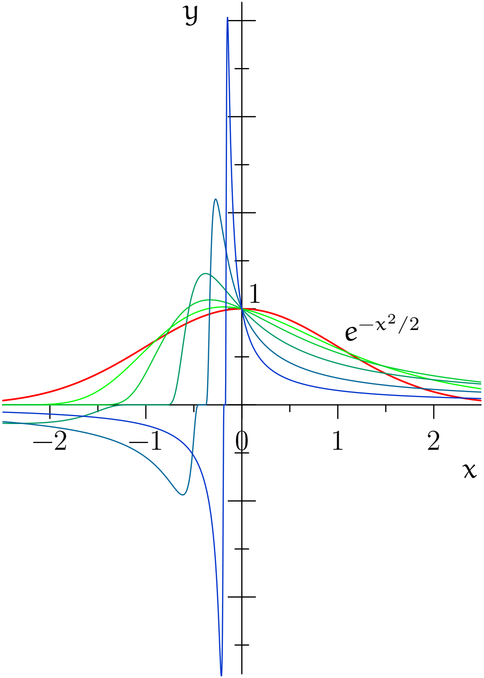
Proof.

1. Intertwining property:

$$\begin{aligned}[\mathcal{W}(\rho(g)v)](h) &= \langle \rho(h^{-1})\rho(g)v, w_0\rangle \\ &= \langle \rho((g^{-1}h)^{-1})v, w_0\rangle \\ &= [\mathcal{W}v](g^{-1}h) \\ &= \Lambda(g)[\mathcal{W}v](h).\end{aligned}$$

2. Since H is invariant under ρ and it is intertwined with Λ by $\mathcal{W}$, the image of $\mathcal{W}$ shall be invariant under Λ.
3. By the above property image of $\mathcal{W}$ is a closed linear span of all left translation of $\mathcal{W}w_0$.

□



If we extend the group from $ax+b$ to $SL_2(\mathbb{R})$ then the same Gaussian as the mother wavelet will produces not only an approximation of the δ-function but approximation δ' distribution as well.

However the extension of the group will not affect the mother wavelet $\frac{1}{x+i}$ it will still generate the Cauchy integral, because is an eigenvector of the subgroup K.

Thus choices of group and mother wavelets produce different frameworks.

Induced Representations

Let G be a group, H its closed subgroup, χ be a linear representation of H in a space V. The set of V-valued functions with the property

$$F(gh) = \chi(h)F(g),$$

is invariant under left shifts.

The restriction of the left regular representation to this space is called an *induced representation*.

Equivalently we consider the *lifting* of $f(x)$, $x \in X = G/H$ to $F(g)$:

$$F(g) = \chi(h)f(p(g)), \qquad p : G \to X, \quad g = s(x)h, \quad p(s(x)) = x.$$

This is a 1-1 map which transform the left regular representation on G to the following action:

$$[\rho'(g)f](x) = \chi(h)f(g \cdot x), \qquad \text{where} \quad gs(x) = s(g \cdot x)h.$$

In the case of $\mathrm{SL}_2(\mathbb{R})$ we have three different types of actions.

Euler's formula, expresses trigonometric functions through the exponent of an imaginary number:

$$e^{it} = \cos t + i \sin t, \qquad \text{with} \qquad |w|^2 = u^2 + v^2. \tag{1}$$

There are other variants of imaginary units, for example $j^2 = 1$. Replacing i by j in (1) we get a key to hyperbolic trigonometry:

$$e^{jt} = \cosh t + j \sinh t, \qquad \text{with} \qquad |w|^2 = u^2 - v^2. \tag{2}$$

For the complete picture we add the parabolic case through the imaginary unit of *dual numbers* defined by $\varepsilon^2 = 0$. Since $\varepsilon^n = 0$ for $n > 1$, Taylor's series imply:

$$e^{\varepsilon t} = 1 + \varepsilon t, \qquad \text{with} \qquad |w|^2 = u^2. \tag{3}$$

Rotation of Wheels

Algebraic

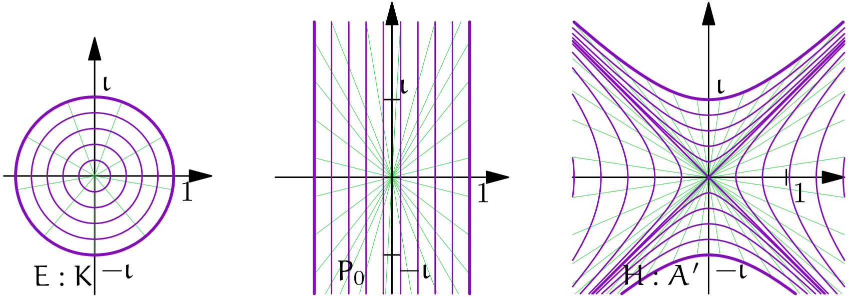


Figure: Rotations of algebraic wheels: elliptic (E), trivial parabolic (P_0) and hyperbolic (H). All blue rims of wheels are defined by the identity $x^2 - \iota^2 y^2 = 1$. Green "spokes" (straight lines from the origin to a point on the rims) are "rotated" by multiplication by $e^{\iota t}$.

Algebraic Expression

of hypercomplex rotations

Elliptic	Parabolic	Hyperbolic
$i^2 = -1$	$\varepsilon^2 = 0$	$j^2 = 1$
$w = x + iy$	$w = x + \varepsilon y$	$w = x + jy$
$\bar{w} = x - iy$	$\bar{w} = x - \varepsilon y$	$\bar{w} = x - jy$
$e^{it} = \cos t + i \sin t$	$e^{\varepsilon t} = 1 + \varepsilon t$	$e^{jt} = \cosh t + j \sinh t$
$\lvert w\rvert_e^2 = w\bar{w} = x^2 + y^2$	$\lvert w\rvert_p^2 = w\bar{w} = x^2$	$\lvert w\rvert_h^2 = w\bar{w} = x^2 - y^2$
$\arg w = \tan^{-1} \frac{y}{x}$	$\arg w = \frac{y}{x}$	$\arg w = \tanh^{-1} \frac{y}{x}$
unit circle $\lvert w\rvert_e^2 = 1$	"unit" strip $x = \pm 1$	unit hyperbola $\lvert w\rvert_h^2 = 1$

Imaginary units are not abstract quantities, they are realised through zero-trace 2×2 matrices:

$$\mathrm{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

with the parabolic ε nicely sitting between the elliptic i and hyperbolic j. Then the matrix multiplication implies $\mathrm{i}^2 = -\mathrm{I}$, $\varepsilon^2 = 0 \cdot \mathrm{I}$, $\mathrm{j}^2 = \mathrm{I}$. Correspondingly we have a matrix form of the Euler's identity identities:

$$\exp\begin{pmatrix} 0 & -\mathrm{t} \\ \mathrm{t} & 0 \end{pmatrix} = \begin{pmatrix} \cos \mathrm{t} & -\sin \mathrm{t} \\ \sin \mathrm{t} & \cos \mathrm{t} \end{pmatrix}, \qquad \exp\begin{pmatrix} 0 & \mathrm{t} \\ \mathrm{t} & 0 \end{pmatrix} = \begin{pmatrix} \cosh \mathrm{t} & \sinh \mathrm{t} \\ \sinh \mathrm{t} & \cosh \mathrm{t} \end{pmatrix}. \tag{5}$$

However the above pattern is only partially reproduced in the matrix form of (3):

$$\exp\begin{pmatrix} 0 & 0 \\ \mathrm{t} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathrm{t} & 1 \end{pmatrix}. \tag{6}$$

A matrix form of Euler's identity is provided by the Cayley transform:

$$\frac{1}{2}\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}=\begin{pmatrix}e^{\mathrm{i}t} & 0\\ 0 & e^{-\mathrm{i}t}\end{pmatrix}, \tag{7}$$

where the Cayley transform maps the upper-half plane to the unit disk. Its hyperbolic cousin is:

$$\frac{1}{2}\begin{pmatrix}1 & \mathrm{j}\\ -\mathrm{j} & 1\end{pmatrix}\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}\begin{pmatrix}1 & -\mathrm{j}\\ \mathrm{j} & 1\end{pmatrix}=\begin{pmatrix}e^{\mathrm{j}t} & 0\\ 0 & e^{-\mathrm{j}t}\end{pmatrix}. \tag{8}$$

In the parabolic case we use the same pattern:

$$\begin{pmatrix}1 & -\varepsilon\\ -\varepsilon & 1\end{pmatrix}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \varepsilon\\ \varepsilon & 1\end{pmatrix}=\begin{pmatrix}1-\varepsilon t & t\\ 0 & 1+\varepsilon t\end{pmatrix}=\begin{pmatrix}e^{-\varepsilon t} & -t\\ 0 & e^{\varepsilon t}\end{pmatrix}. \tag{9}$$

However the complete harmony is spoilt by the off-diagonal term.

Characters from Möbius Maps

The matrix Euler's identity folds back to numbers through the Möbius maps, diagonal matrices act by multiplications by e^{2it} and e^{2jt}:

$$\begin{aligned} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} : \quad -i \quad &\mapsto \sin 2t - i\cos 2t; \\ \begin{pmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{pmatrix} : \quad -j \quad &\mapsto -\sinh 2t - j\cosh 2t. \end{aligned}$$

However the parabolic action is not such a simple one:

$$\begin{pmatrix} e^{-\varepsilon t} & 0 \\ t & e^{\varepsilon t} \end{pmatrix} : -\varepsilon \mapsto \frac{1}{t} + \varepsilon\left(\frac{1}{t^2} - 1\right). \tag{10}$$

The parabolic "unit circle" is defined by the equation $x^2 - y = 1$ (c.f. previous $x^2 = 0$).

Rotation of Wheels, Geometric

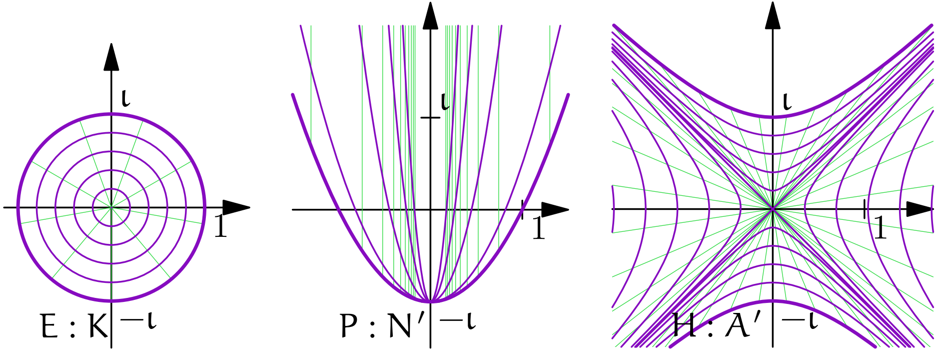


Figure: Rotation of geometric wheels: elliptic (E), parabolic P' and hyperbolic (H). Blue orbits are level lines for the respective moduli. Green straight lines join points with the same value of argument and are drawn with the constant "angular step" in each case.

Modulus and Argument

In the elliptic and hyperbolic cases orbits of rotations are points with the constant norm (modulus): either $x^2 + y^2$ or $x^2 - y^2$.

Definition

Orbits of parabolic rotations are contour lines for moduli (norms):

$$|u + \varepsilon v| = \frac{u^2}{v+1}. \tag{11}$$

The only straight lines preserved by both the parabolic rotations are vertical lines, thus we will treat them as "spokes" for parabolic wheels, or "points on the complex plane with the same argument":

Definition

Parabolic arguments are defined as follows:

$$\arg(u + \varepsilon v) = \frac{1}{u}. \tag{12}$$

Parabolic multiplication

Modulus and argument behave naturally under rotations:

Proposition

Let w_s *is a parabolic rotation of* w *by* $s = t^{-1}$. *Then:*

$$|w_s| = |w|, \qquad \arg w_s = \arg w + s.$$

Thus we revert theorems into definitions to assign multiplication.

Definition

The product of w_1 and w_2 is uniquely defined by:

1. $\arg(w_1 w_2) = \arg w_1 + \arg w_2$;
2. $|w_1 w_2| = |w_1| \cdot |w_2|$.

We also need a special form of parabolic conjugation.

Definition

Parabolic conjugation is given by $\overline{u + \varepsilon v} = -u + \varepsilon v$.

Multiplication by a scalar is straightforward: it should preserve the argument and scale the norm of vectors:

$$a \cdot (u, v) \quad = \quad \left(u, \frac{v+1}{a} - 1\right). \tag{13}$$

Let us introduce the lexicographic order on $\mathbb{R}^2$:

$$(u, v) \prec (u', v') \quad \text{if and only if} \quad \begin{cases} \text{either} & u < u'; \\ \text{or} & u = u',\ v < v'. \end{cases}$$

One can define functions $\min$ and $\max$ of a pair of points on $\mathbb{R}^2$ correspondingly. An addition of two vectors can be defined either as their minimum or maximum. A similar definition is used in *tropical mathematics*, also known as Maslov dequantisation or $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$ algebras. It is easy to check that such an addition is distributive with respect to scalar multiplication and consequently is invariant under parabolic rotations.

Definition

Parabolic addition of vectors is defined by the following formulae:

$$\arg(w_1 + w_2) = \frac{\arg w_1 \cdot |w_1| + \arg w_2 \cdot |w_2|}{|w_1 + w_2|}, \tag{14}$$

$$|w_1 + w_2| = |w_1| + |w_2|. \tag{15}$$

The rule for the norm of sum (15) looks too trivial, however it nicely sits in between the elliptic $|w + w'| \leqslant |w| + |w'|$ and hyperbolic $|w + w'| \geqslant |w| + |w'|$ inequalities for norms.
Both formulae (14)–(15) together uniquely define explicit expressions for addition of vectors. Although those expressions are rather cumbersome and not really much needed.

We get an algebra!

Proposition

Vector addition satisfy the following conditions:

1. *They are commutative and associative.*
2. *They are distributive for multiplication;*
3. *They are parabolic rotationally invariant;*
4. *They are distributive in both ways for the scalar multiplication:*

$$a \cdot (w_1 + w_2) = a \cdot w_1 + a \cdot w_2, \quad (a + b) \cdot w = a \cdot w + b \cdot w.$$

Proposition

The zero vector is $(\infty, -1)$ *and consequently the inverse of* (u, v) *is* $(u, -v - 2)$.

Birational geometry

Basis representation: $(u,v) = a \cdot (1,0) + b \cdot (-1,0)$ implies:

$$a = \frac{u^2 - v}{2}(1+u), \qquad b = \frac{u^2 - v}{2}(1-u).$$

To transfer parabolic rotations from (u,v)-plane to (a,b)-coordinates:

$$\begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix} : (a,b) \mapsto \left(a + \frac{t}{2}(a+b), b - \frac{t}{2}(a+b) \right).$$

After euclidean rotation by 45° given by $(a,b) \mapsto (a+b, a-b)$ this coincides with the "rectangular" parabolic rotation. Moreover:

Proposition

The above transformations maps algebraic operations in the exotic form to corresponding operations on dual numbers.

No surprise: any associative and commutative 2D algebra is isomorphic to either complex, dual or double numbers. But presence of singularities is the subject of birational geometry.

Affine Group

For $G = SL_2(\mathbb{R})$ and $H = F$ the action on G/H is:

$$g : u \mapsto p(g^{-1} * s(u)) = \frac{au+b}{cu+d}, \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We calculate also that

$$r(g^{-1} * s(u)) = \begin{pmatrix} (cu+d)^{-1} & 0 \\ c & cu+d \end{pmatrix}.$$

A generic character of F is a power of its diagonal element:

$$\rho_\kappa \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^\kappa.$$

Thus the corresponding realisation of induced representation is:

$$\rho_\kappa(g) : f(u) \mapsto \frac{1}{(cu+d)^\kappa} f\left(\frac{au+b}{cu+d}\right) \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Induced Wavelet Transform

Let $v_0 \in \mathcal{H}$ be an eigenfunction as follows:

$$\rho(h)v_0 = \tilde{\chi}(h) \cdot v_0, \qquad \text{for all } h \in \tilde{H}.$$

It is suitable to be the *mother wavelet* (*vacuum vector*). Then we have

$$\begin{aligned} [\mathcal{W}f](gh) &= \langle f, \rho(gh)v_0 \rangle = \langle f, \rho(g)\rho(h)v_0 \rangle \\ &= \langle f, \tilde{\chi}(h) \cdot \rho(g)v_0 \rangle = \tilde{\chi}(h^{-1}) \langle f, \rho(g)v_0 \rangle . \end{aligned}$$

For v_0 the *induced wavelet transform* $\mathcal{W} : \mathcal{H} \to L_\infty(G/\tilde{H})$ by

$$[\mathcal{W}f](w) = \langle f, \rho_0(s(w))v_0 \rangle , \qquad (16)$$

where $w \in G/\tilde{H}$ and $s : G/\tilde{H} \to G$.
It intertwines ρ with a representation induced by $\tilde{\chi}^{-1}$ of $\tilde{H}$.
Particularly, it intertwines ρ with the representation associated to G-action on the homogeneous space $G/\tilde{H}$.

The Lie algebra $\mathfrak{sl}_2$ of $\mathrm{SL}_2(\mathbb{R})$ consists of all 2×2 real matrices of trace zero. One can introduce a basis:

$$A = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The commutator relations are

$$[Z, A] = 2B, \qquad [Z, B] = -2A, \qquad [A, B] = -\frac{1}{2}Z.$$

The derived representation for a vector field $Y \in \mathfrak{sl}_2$ is defined through the exponential map $\exp : \mathfrak{sl}_2 \to \mathrm{SL}_2(\mathbb{R})$ by the standard formula:

$$d\rho^Y = \frac{d}{dt}\rho(e^{tY})\Big|_{t=0}. \tag{17}$$

Example

For $A = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we get $e^{tA} = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$. Thus

$$d\rho^A = \frac{d}{dt}\rho(e^{tA})\Big|_{t=0} = \frac{d}{dt}\left[\frac{1}{e^{\kappa t/2}} f\left(e^{-t}u\right)\right]\Big|_{t=0} = -\frac{\kappa}{2}f(u) - uf'(u).$$

On the real line the derived representation is represented by the vector fields:

$$d\rho_\kappa^A = -\frac{\kappa}{2}\cdot I - u\partial_u, \tag{18}$$

$$d\rho_\kappa^B = \frac{\kappa}{2}u\cdot I + \frac{1}{2}(u^2-1)\partial_u, \tag{19}$$

$$d\rho_\kappa^Z = -\kappa u\cdot I - (u^2+1)\partial_u. \tag{20}$$

Let ρ be a unitary representation of Lie group G with the derived representation $d\rho$ of $\mathfrak{g}$. Let a mother wavelet w_0 be a null-solution, i.e. $Aw_0 = 0$, for the operator $A = \sum_J a_j d\rho^{X_j}$, where $X_j \in \mathfrak{g}$. Then the wavelet transform $F(g) = \mathcal{W}f(g) = \langle f, \rho(g)w_0 \rangle$ for any f satisfies to:

$$DF(g) = 0, \qquad \text{where} \quad D = \sum_j a_j \mathfrak{L}^{X_j}.$$

Here $\mathfrak{L}^{X_j}$ are left the invariant fields (Lie derivatives) on G corresponding to X_j.

If $\mathfrak{L}^{X_j}$ is derived representation of Lie derivative A, N, K (without the matching subgroup) then C-R operator and Laplacian are given by:

$$D = \iota \mathfrak{L}^{A} + \mathfrak{L}^{X}, \qquad \text{and} \qquad \Delta = D\bar{D} = -\sigma {\mathfrak{L}^{A}}^2 + {\mathfrak{L}^{X}}^2, \tag{21}$$

where X is in the orthogonal complement (with respect to the Killing form) of the corresponding subgroup K, N, A.

Consider the representation ρ

$$\rho_2(g) : f(u) \mapsto \frac{1}{(cu+d)^2} f\left(\frac{au+b}{cu+d}\right) \qquad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let A and $N \in \mathfrak{sl}_2$ generates $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ and $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then the derived representations are:

$$[d\rho^A f](x) = f(x) + xf'(x), \qquad [d\rho^N f](x) = f'(x).$$

The corresponding left invariant vector fields on upper half-plane are:

$$\mathfrak{L}^A = a\partial_a, \qquad \mathfrak{L}^N = a\partial_b.$$

The mother wavelet $\frac{1}{x+i}$ is a null solution of the operator $d\rho^A + i d\rho^N = I + (x+i)\frac{d}{dx}$. Therefore the wavelet transform will consist of the null solutions to the operator $\mathfrak{L}^A - i\mathfrak{L}^N = a(\partial_a + i\partial_b)$—the Cauchy-Riemann operator.

The infinitesimal version of the eigenvector property $\rho(h)v_0 = \chi(h) \cdot v_0$ is $d\rho_n^Z v_0 = \lambda v_0$, explicitly, cf. (20)

$$nuf(u) + f'(u)(1 + u^2) = \lambda f(u).$$

The generic solution is:

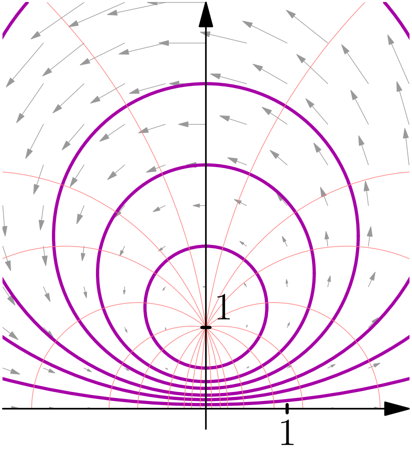
$$f(u) = \frac{1}{(1 + u^2)^{n/2}} \left(\frac{u + i}{u - i}\right)^{i\lambda/2} = \frac{(u + i)^{(i\lambda - n)/2}}{(u - i)^{(i\lambda + n)/2}}.$$

To avoid multivalent function we need to put $\lambda = im$ with an integer m. The Cauchy–Riemann condition (which turn to be later the same as "the minimal weight condition") suggests $m = n$. Thus, the induced wavelet transform is:

$$\hat{f}(x, y) = \langle f, \rho_n f_0 \rangle = \int_{\mathbb{R}} f(u) \, \frac{\sqrt{y}}{u - x - iy} \, dx = \sqrt{y} \int_{\mathbb{R}} f(u) \, \frac{dx}{u - (x + iy)}$$

And its image consists of null solutions of Cauchy–Riemann type equations. For $m > n$ we obtain *polyanalytic* functions annihilated by powers of Cauchy–Riemann operator.

Fix Subgroups of i and j



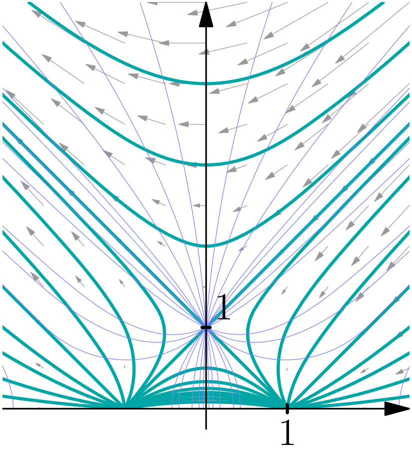


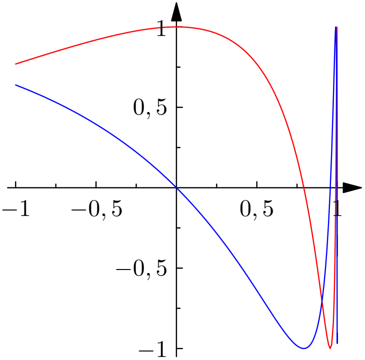
Figure: Elliptic and hyperbolic fix groups of the imaginary units.
In the hyperbolic case there are fixed geometric sets: $\{-1, 1\}$, $(-1, 1)$, $\mathbb{R}$.

For the subgroup A' generated by $B \in \mathfrak{sl}_2$ the derived representation, cf. (19):

$$d\rho_n^B f(u) = -nuf(u) + (u^2-1)f'(u).$$

It has two singular point ± 1, its solution has compact support $[-1,1]$.

$$\begin{aligned} f(x) &= \frac{1}{(u^2-1)^{n/2}}\left(\frac{u+1}{u-1}\right)^{\lambda/2} \\ &= \frac{(u+1)^{(\lambda-n)/2}}{(u-1)^{(\lambda+n)/2}}. \end{aligned}$$



For $\lambda = jm$ we also get, cf. K-case:

$$f(x) = \frac{(x+j)^{(m-\kappa)/2}}{(x-j)^{(m+\kappa)/2}}.$$

Hyperbolic Wavelets from Double Numbers

The choice of the A-eigenvector as mother wavelet:

- $f_0 = \delta(x \pm 1)$—Dirichlet condition.
- $f_0 = \dfrac{1}{(x-j)^\sigma} = \left(\dfrac{x+j}{x^2-1}\right)^\sigma$—Neumann condition.
- $f_0 = \dfrac{\chi(1-x^2)}{(x-j)^\sigma}$—space-like and time-like separation, Fig. 3.
- ...(combination of above)

Then we follow the general scheme both for wavelets with complex and double valued wavelets:

- *wavelets* or *coherent states* $v_\sigma(g,z) = \rho_\sigma(g)v_0(z)$.
- d'Alambert integral from the universal *wavelet transforms*

$$\mathcal{W}_\sigma : f(z) \mapsto \mathcal{W}_\sigma f(u) = \langle f(z), \rho_\sigma v_0(u,z)\rangle$$

The subgroup N consists of shifts, the eigenfunction is $e^{\lambda u}$ and the induced wavelet transform coincides with the Fourier transform.
For the subgroup N', the generator is $d\rho_n^{Z/2-B} = (un)\cdot I - u^2\cdot\partial_u$, cf. (19–20). The eigenvector $d\rho_n^{Z/2-B} f = \lambda f$ is $f_0(u) = u^n e^{\frac{\lambda}{u}}$.
Consider some identities for dual numbers:

$$e^{\varepsilon\alpha t} = 1+\varepsilon\alpha t; \qquad (t\pm\varepsilon)^\alpha = t^{\alpha-1}(t\pm\varepsilon\alpha); \qquad (t-\varepsilon)(t+\varepsilon) = t^2.$$

Combining them together we can write for $\lambda = \varepsilon m$:

$$e^{\frac{\varepsilon m}{u}} = 1+\frac{\varepsilon m}{u} = \left(\frac{u+\varepsilon}{u-\varepsilon}\right)^{m/2}$$

Then the solution $f_0(u) = u^n e^{\frac{\lambda}{u}}$ is:

$$|u|^{-\kappa} e^{-\frac{\varepsilon m}{u}} = \frac{1}{((u+\varepsilon)(u-\varepsilon))^{\kappa/2}}\left(\frac{u+\varepsilon}{u-\varepsilon}\right)^{m/2} = \frac{(u+\varepsilon)^{(m-\kappa)/2}}{(u-\varepsilon)^{(m+\kappa)/2}} \quad (22)$$

The respective wavelet transform is again very similar to the complex case.

Expansion over Eigenfunctions

Wavelet are decomposable $v_g(x) = \sum_\lambda c_\lambda(x)\phi_\lambda(g)$ over the *complete set* of its eigenfunctions $\phi_\alpha(u)$ of the subgroup $\tilde{H}$. Then from the wavelet transform:

$$\langle f(z), v_g(z)\rangle = \left\langle f(z), \sum_\lambda c_\lambda(z)\phi_\lambda(g) \right\rangle = \sum_\lambda \phi_\lambda(g) \langle f(z), c_\lambda(z)\rangle$$

In the elliptic case eigenvectors of K are

$$f_{m,\kappa}(u) = \frac{(u-i)^{(m-\kappa)/2}}{(u+i)^{(m+\kappa)/2}}.$$

If we make an Cayley transform to the unit disk those function become z^m, $m = 0, 1, 2, \ldots$ and the decomposition is the *Taylor series*:

$$f(z) = \sum_0^\infty c_n z^n.$$

Raising/Lowering Operators

Denote $\tilde{X} = d\rho(X)$ for $X \in \mathfrak{sl}_2$. Let $X = Z$ be the generator of the compact subgroup K, eigenspaces $\tilde{Z}v_k = ikv_k$ are parametrised by an integer $k \in \mathbb{Z}$. The raising/lowering operators $L_\pm$:

$$[\tilde{Z}, L_\pm] = \lambda_\pm L_\pm. \tag{23}$$

[$L_\pm$ are eigenvectors for operators $\mathrm{ad}\, Z$ of adjoint representation of $\mathfrak{sl}_2$.] From the commutators (23) L_+v_k are eigenvectors of $\tilde{Z}$ as well:

$$\begin{aligned}\tilde{Z}(L_+v_k) &= (L_+\tilde{Z} + \lambda_+ L_+)v_k = L_+(\tilde{Z}v_k) + \lambda_+ L_+ v_k \\ &= ikL_+v_k + \lambda_+ L_+ v_k = (ik + \lambda_+)L_+v_k.\end{aligned}$$

Thus those operators acts on a chain of eigenspaces:

$$\ldots \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{ik-\lambda} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{ik} \underset{L_-}{\overset{L_+}{\rightleftarrows}} V_{ik+\lambda} \underset{L_-}{\overset{L_+}{\rightleftarrows}} \ldots$$

Finding Raising/Lowering Operators

Elliptic and hyperbolic

Subgroup K. Assuming $L_+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ we obtain a linear equation:

$$c = 0, \qquad 2a = \lambda_+ b, \qquad -2b = \lambda_+ a.$$

The equations have a solution if and only if $\lambda_+^2 + 4 = 0$, and the raising operator is $L_+ = i\tilde{A} + \tilde{B}$.

Subgroup A. For the commutator $[\tilde{B}, L_+] = \lambda L_+$ we will got the system:

$$2c = \lambda a, \qquad b = 0, \qquad \frac{a}{2} = \lambda c.$$

A solution exists if and only if $\lambda^2 = 1$. The obvious values $\lambda = \pm 1$ with the operator $L_\pm = \pm\tilde{A} + \tilde{Z}/2$. Each indecomposable $\mathfrak{sl}_2$-module is formed by one-dimensional chain of eigenvalue with transitive action of raising/lowering operators.

Hyperbolic Ladder Operators

Double numbers: $\lambda = \pm \mathrm{j}$ solves $\lambda^2 = 1$ additionally to $\lambda = \pm 1$.
The raising/lowering operators $L^h_\pm = \pm \mathrm{j}\tilde{A} + \tilde{Z}/2$ "orthogonal" to $L_\pm$.

$$
\begin{tikzcd}
& \cdots \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & \cdots \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & \cdots \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & \\
\cdots \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] & V_{(n-2)+j(k-2)} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & V_{n+j(k-2)} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & V_{(n+2)+j(k-2)} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & \cdots \\
\cdots \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] & V_{(n-2)+jk} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & V_{n+jk} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & V_{(n+2)+jk} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & \cdots \\
\cdots \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] & V_{(n-2)+j(k+2)} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & V_{n+j(k+2)} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & V_{(n+2)+j(k+2)} \arrow[r, shift left, "L_h^+"] \arrow[r, leftarrow, shift right, "L_h^-"'] \arrow[d, shift left, "L_j^+"] \arrow[d, leftarrow, shift right, "L_j^-"'] & \cdots \\
& \cdots & \cdots & \cdots &
\end{tikzcd}
$$

Finding Raising/Lowering Operators

Parabolic

A generator $X = -B + Z/2$ of the subgroup N' gets the equations:

$$b + 2c = \lambda a, \qquad -a = \lambda b, \qquad \frac{a}{2} = \lambda c,$$

which can be resolved if and only if $\lambda^2 = 0$. Restricted with the real (complex) root $\lambda = 0$ make operators $L_\pm = -\tilde{B} + \tilde{Z}/2$. Does not affect eigenvalues and thus are useless. However, a dual number $\lambda_t = t\varepsilon$, $t \in \mathbb{R}$ leads to the operator $L_\pm = \pm t\varepsilon\tilde{A} - \tilde{B} + \tilde{B}/2$, which allow us to build a $\mathfrak{sl}_2$-modules with a one-dimensional continuous(!) chain of eigenvalues.

- K Introduction of complex numbers is a necessity for the *existence* of raising/lowering operators;
- N we need dual numbers to make raising/lowering operators *useful*;
- $\mathcal{A}$ double number are required for neither existence nor usability of raising/lowering operators, but do provide an enhancement.

Principle of Similarity and correspondence

1. Subgroups K, N and A play the similar role in a structure of the group $SL_2(\mathbb{R})$ and its representations.
2. The subgroups shall be swapped together with the respective replacement of hypercomplex unit ι.

Manifestations:

- The action of $SL_2(\mathbb{R})$ on $SL_2(\mathbb{R})/H$ for $H = A'$, N' or K and linear-fractional transformations of respective numbers.
- Subgroups K, N' and A' and unitary rotations of respective unit cycles.
- Representations induced from subgroup K, N' or A' and unitarity in respective numbers.
- The connection between raising/lowering operators for subgroups K, N' or A' and corresponding numbers.