Geodesic Mappings and Einstein Spaces

Josef Mikeš and Irena Hinterleitner

Palacky University Olomouc, Brno University of Technology

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Topics

1. Geodesic mapping theory for $V_n \to \bar{V}_n$ of class C^1
2. Geodesic mapping theory for $V_n \to \bar{V}_n$ of class C^2
3. Geodesic mapping between $V_n \in C^r$ $(r > 2)$ and $\bar{V}_n \in C^1$
4. On geodesic mappings of Einstein spaces

Assume the (pseudo-) Riemannian manifolds $V_n = (M, g, \nabla)$ and $\bar{V}_n = (\bar{M}, \bar{g}, \bar{\nabla})$. Here V_n, $\bar{V}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Definition

A diffeomorphism $f\colon\ V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

A manifold V_n admits a geodesic mapping onto $\bar{V}_n$ if and only if the *Levi-Civita equations*

$$(1) \qquad \bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X$$

hold for any tangent fields X, Y and where ψ is a differential form.
If $\psi \equiv 0$ than f is *affine* or *trivially geodesic*.
In local form: $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i$,
where $\Gamma^h_{ij}(\bar{\Gamma}^h_{ij})$ are the Christoffel symbols of V_n and $\bar{V}_n$,
ψ_i are components of ψ and δ^h_i is the Kronecker delta.

Equations (1) are equivalent to the following equations

$$(2) \qquad \bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}$$

where "," denotes the covariant derivative on V_n. It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+1)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad \partial_i = \partial / \partial x^i.$$

Sinyukov proved that the Levi-Civita equations are equivalent to

$$(3) \qquad a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

where

$$(4) \quad \text{(a)} \ \ a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \text{(b)} \ \ \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha.$$

From (3) follows $\lambda_i = \partial_i \lambda = \partial_i(\frac{1}{2} a_{\alpha\beta} g^{\alpha\beta})$. On the other hand

$$(5) \quad \bar{g}_{ij} = \mathrm{e}^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \|\tilde{g}_{ij}\| = \|g^{i\alpha} g^{j\beta} a_{\alpha\beta}\|^{-1}.$$

The above formulas are the criterion for geodesic mappings $V_n \to \bar{V}_n$ globally as well as locally.

2.Geodesic mapping theory for $V_n \to \bar{V}_n$ of class C^2

Let V_n and $\bar{V}_n \in C^2$, then for geodesic mappings $V_n \to \bar{V}_n$ the Riemann and the Ricci tensors transform in this way

(6)

(a) $\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}$; (b) $\bar{R}_{ij} = R_{ij} - (n-1)\psi_{ij}$,

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$,
and the Weyl tensor of projective curvature, which is defined in the following form

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{n-1}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik}\right),$$

is invariant.

The integrability conditions of the Sinyukov equations (3)

have the following form

$$(7)\qquad a_{i\alpha}R^{\alpha}_{jkl} + a_{j\alpha}R^{\alpha}_{ikl} = g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k}.$$

After contraction with g^{jk} we get

$$(8)\qquad n\lambda_{i,l} = \mu g_{il} - a_{i\alpha}R^{\alpha}_{l} + a_{\alpha\beta}R^{\alpha}{}_{il}{}^{\beta}$$

where $R^{\alpha}{}_{il}{}^{\beta} = g^{\beta k}R^{\alpha}{}_{ilk}$; $R^{\alpha}_{l} = g^{\alpha j}R_{jl}$ and $\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}$.

3. Geodesic mapping between $V_n \in C^r$ $(r > 2)$ and $\bar{V}_n \in C^1$

Theorem 1

If $V_n \in C^r$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^1$, then $\bar{V}_n \in C^r$.

This Theorem is more strong than following theorem

Theorem 2

If $V_n \in C^r$ $(r > 2)$ admits geodesic mappings onto $\bar{V}_n \in C^2$, then $\bar{V}_n \in C^r$.

Lemma 1

Let $\lambda^h \in C^1$ be a vector field and ϱ a function.
If $\partial_i \lambda^h - \varrho\, \delta_i^h \in C^1$ then $\lambda^h \in C^2$ and $\varrho \in C^1$.

Sketch of the proof:

The condition $\partial_i\lambda^h - \varrho\,\delta_i^h \in C^1$ can be written in the following form

$$\partial_i\lambda^h - \varrho\delta_i^h = f_i^h(x), \tag{9}$$

where $f_i^h(x)$ are functions of class C^1. Evidently, $\varrho \in C^0$. For fixed but arbitrary indices $h \neq i$ we integrate (9) with respect to dx^i:

$$\lambda^h = \Lambda^h + \int_{x_o^i}^{x^i} f_i^h(x^1, \ldots, x^{i-1}, t, x^{i+1}, \ldots, x^n)\, dt,$$

where Λ^h is a function, which does not depend on x^i.
Because of the existence of the partial derivatives of the functions λ^h and the above integrals, also the derivatives $\partial_h\Lambda^h$ exist.

Then we can write (9) for $h = i$:

$$(10)\quad \varrho = -f_h^h + \partial_h \Lambda^h + \int_{x_o^i}^{x^i} \partial_h f_i^h(x^1, \dots, x^{i-1}, t, x^{i+1}, \dots, x^n)\, dt.$$

Because the derivative with respect to x^i of the right-hand side of (10) exists, the derivative of the function ϱ exists, too. Obviously $\partial_i \varrho = \partial_h f_i^h - \partial_i f_h^h$, therefore $\varrho \in C^1$ and from (9) follows $\lambda^h \in C^2$.

In a similar way we can prove the following: *if* $\lambda^h \in C^r$ $(r \geq 1)$ *and* $\partial_i \lambda^h - \varrho \delta_i^h \in C^r$ *then* $\lambda^h \in C^{r+1}$ *and* $\varrho \in C^r$.

Lemma 2

If $V_n \in C^3$ *admits a geodesic mapping onto* $\bar{V}_n \in C^2$, *then* $\bar{V}_n \in C^3$.

Skach of the proof

In this case Sinyukov's equations (3) *and* (8) *hold. According to the assumptions* $g_{ij} \in C^3$ *and* $\bar{g}_{ij} \in C^2$. *By a simple check-up we find* $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ *and*
$R^h_{ijk}, R^h{}_{ij}{}^k, R_{ij}, R^h_i \in C^1$.
From the above-mentioned conditions we easily convince ourselves that we can write equation (8) *in the form* (9), *where*
$\lambda^h = g^{h\alpha}\lambda_\alpha \in C^1$, $\varrho = \mu/n$ *and*
$f_i^h = (-\lambda^\alpha \Gamma^h_{\alpha i} - g^{h\gamma} a_{\alpha\gamma} R^\alpha_i + g^{h\gamma} a_{\alpha\beta} R^\alpha{}_{i\gamma}{}^\beta)/n \in C^1$.
From Lemma 1 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, *and evidently* $\lambda_i \in C^2$. *Differentiating* (3) *twice we convince ourselves that* $a_{ij} \in C^3$. *From this and formula* (5) *follows that also* $\Psi \in C^3$ *and* $\bar{g}_{ij} \in C^3$.

Further we notice that for geodesic mappings between V_n and $\bar{V}_n$ of class C^3 holds the third set of Sinyukov equations:

$$(11) \qquad (n-1)\mu_{,k} = 2(n+1)\lambda_\alpha R^\alpha_k + a_{\alpha\beta}(2R^\alpha{}_{k,}{}^\beta - R^{\alpha\beta}{}_{,k}).$$

If $V_n \in C^r$ and $\bar{V}_n \in C^2$, then by Lemma 2, $\bar{V}_n \in C^3$ and (11) hold. Because Sinyukov's system (3), (8) and (11) is closed, we can differentiate equations (3) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ $(\equiv \bar{V}_n \in C^r)$.

Remark

Because for holomorphically projective mappings of Kähler (and also hyperbolic and parabolic Kähler) spaces hold equations analogical to (3) and (8), from Lemma 1 follows an analog to Theorem 1 for these mappings.

Einstein spaces V_n are characterized by the condition

$$Ric = \mathrm{const} \cdot g,$$

so $V_n \in C^2$ would be sufficient.

We remark that spaces of constant curvature are Einstein spaces and Einstein spaces V_3 are always have constant curvature. Therefore many properties of Einstein spaces appear when

$$V \in C^3 \quad \text{and} \quad n > 3.$$

Moreover, it is known (D.M. DeTurck and J.L. Kazdan) that an Einstein space V_n belongs to C^ω, i.e., for all points of V_n, there exists local coordinate system x for which $g_{ij}(x) \in C^\omega$ (analytic coordinate system).

It is known that Riemannian spaces of constant curvature form a closed class with respect to geodesic mappings (Beltrami theorem).

Theorem 3

If the Einstein space V_n admits a nontrivial geodesic mapping onto a (pseudo-) Riemannian space $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.

In 1978 in the PhD thesis Mikeš proved that above Theorem holds locally for $V_n \in C^3$ and $\bar{V}_n \in C^3$.
From Theorem 2 this Theorem holds for $V_n \in C^3$ and $\bar{V}_n \in C^1$.
Moreover from results by DeTurck this Theorem holds GLOBALLY and exists common coordinate system in which $V_n \in C^\omega$ and $\bar{V}_n \in C^\omega$.

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Thank you for your attention!