Trajectories of the Plate-Ball Problem

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Outline

The Problem of a Rolling Sphere (David Kendall, Oxford, 1950s)

- A spherical ball rests on an infinite horizontal table.
- The ball has to be transferred from a given initial state to an arbitrary final state, meaning its position and orientation in space via a sequence of moves.
- Each move consists of rolling the ball along some straight line on the table.
- The axis of rotation must be horizontal and there must be no slipping between the ball and the table.

The Problem of a Rolling Sphere
(David Kendall, Oxford, 1950s)

How many moves, N, will be necessary and sufficient to reach any final state?

Ans: $N = 3$ (Hammersley, 1983)

Rolling Along a Curve of Shortest Length
(Hammersley, 1983)

Which is the shortest curved path between the prescribed initial and final states?

Quaternions and Rotations

A quaternion, q, is a combination of a scalar and a vector

$$q = \rho_0 + \rho_1 \mathbf{i} + \rho_2 \mathbf{j} + \rho_3 \mathbf{k}$$

A quaternion product, $\mathbf{v}\,\mathbf{w}$, of two vectors $\mathbf{v}$ and $\mathbf{w}$ is a quaternion

$$\mathbf{v}\,\mathbf{w} = -\,\mathbf{v}\cdot\mathbf{w} + \mathbf{v}\times\mathbf{w}$$

A rotation through angle φ about the unit vector $\mathbf{u}$ is represented by the quaternion pair

$$\pm q = \pm e^{\frac{1}{2}\varphi\mathbf{u}} = \pm(\cos\frac{1}{2}\varphi + \mathbf{u}\sin\frac{1}{2}\varphi) = \pm\mathbf{u}_2\mathbf{u}_1$$

Optimal Control Problem

State equation: $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{h})$, $\mathbf{x} \in R^n$, $\mathbf{h} \in \mathbb{R}^r$

Control function: $\mathbf{h} = \mathbf{h}(t) \in \Delta$ (admissible controls)

Trajectory of the controlled system: $\mathbf{x} = \mathbf{x}(t)$

Cost function: $\mathcal{J}(\mathbf{h}) = \int_0^T f_0(t, \mathbf{x}(\mathbf{t}), \mathbf{h}(t))\mathrm{d}t$

Given an initial point $\mathbf{x}(0) = \mathbf{x}_0$ and a final point $\mathbf{x}(T) = \mathbf{x}_E$, find optimal control $\mathbf{h}^\star(t)$, such that minimizes the cost function

$$\mathcal{J}(\mathbf{h}^\star) \leq \mathcal{J}(\mathbf{h})$$

Pontryagin Maximum Principle (free-final-time problem)

Optimal control $\mathbf{h}^\star(t)$ and optimal trajectory $\mathbf{x}^\star(t)$ are known. Hence, there exists adjoint function $\lambda^\star(t) \in \mathbb{R}^{n+1}$ satisfying

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}}(\lambda(t), \mathbf{x}^\star(t), \mathbf{h}^\star(t))$$

where the Hamiltonian is given by

$$\mathcal{H}(\lambda, \mathbf{x}, \mathbf{h}) = \sum_{\alpha=0}^{n} \lambda_\alpha f_\alpha(\mathbf{x}, \mathbf{h})$$

and the Pontryagin maximum principle holds

$$\max_{h \in \Delta} \mathcal{H}(\lambda, \mathbf{x}^\star, \mathbf{h}) = \mathcal{H}(\lambda, \mathbf{x}^\star, \mathbf{h}^\star) \equiv 0, \quad \lambda_0 \leq 0$$

Differential Equation of the Rolling Sphere (Hammersley, 1983)

The sphere is rolled along a unit speed curve Γ, parameterized by the arc length t. The resultant rotation at t is given by the quaternion

$$q(t) = \rho_0(t) + \rho_1(t)\,\mathbf{i} + \rho_2(t)\,\mathbf{j} + \rho_3(t)\,\mathbf{k}$$

The instantaneous axis of rotation is specified by the unit vector $h(t)$ along the axis of rotation

$$\mathbf{h}(t) = h_1(t)\,\mathbf{i} + h_2(t)\,\mathbf{j}$$

The differential equation of the rolling sphere is of the form:

$$\dot{q} = \frac{1}{2}\mathbf{h}\,q$$

Rolling a Sphere Along a Curve of the Shortest Length
(a "minimum-time" optimal control problem)

Cost function: $\mathcal{J} \equiv T = \int_0^T 1.\mathrm{d}t$ (T – length of the curve)

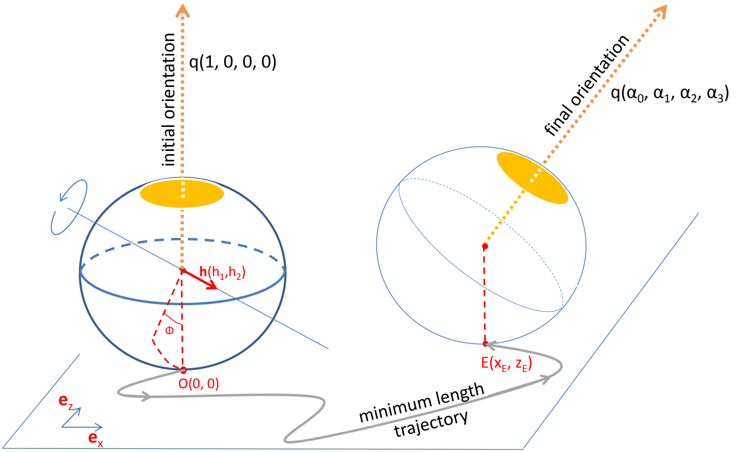
Control function: $\mathbf{h} = \mathbf{h}(t) \in \Delta$ $(\Delta = \{\mathbf{h} : |\mathbf{h}| = 1\})$

Initial position and orientation: $O(0, 0)$, $q(0) = (1, 0, 0, 0)$

Final position:

$$x(T) \equiv \int_0^T h_2(t)\,\mathrm{d}t = x_E, \qquad z(T) \equiv \int_0^T h_1(t)\,\mathrm{d}t = -z_E$$

Final orientation: $q(T) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$



initial orientation
q(1, 0, 0, 0)
final orientation
q(α0, α1, α2, α3)
h(h1,h2)
Φ
O(0, 0)
E(xE, zE)
ez
ex
minimum length
trajectory

State Variables and State Equations

State variables: $\rho = (\rho_0,\ \rho_1,\ \rho_2,\ \rho_3,\ \rho_{-1},\ \rho_4,\ \rho_5)$

$(\rho_0,\ \rho_1,\ \rho_2,\ \rho_3)$ – coordinates of the quaternion q

$(\rho_{-1},\ \rho_4,\ \rho_5)$ – state variable inferred from the integrals

$$\int_0^T 1\,\mathrm{d}t = T, \qquad \int_0^T h_2(t)\,\mathrm{d}t = x_E, \qquad \int_0^T h_1(t)\,\mathrm{d}t = -z_E$$

State equations:

$$\dot\rho_0 = -\frac{1}{2}(h_1\rho_1 + h_2\rho_2), \qquad \dot\rho_1 = \frac{1}{2}(h_1\rho_0 + h_2\rho_3)$$

$$\dot\rho_2 = \frac{1}{2}(-h_1\rho_3 + h_2\rho_0), \qquad \dot\rho_3 = \frac{1}{2}(h_1\rho_2 - h_2\rho_1)$$

$$\dot\rho_{-1} = 1, \qquad \dot\rho_4 = \frac{1}{2}h_1, \qquad \dot\rho_5 = \frac{1}{2}h_2$$

Adjoint Variables and the Hamiltonian (Arthurs and Walsh, 1986)

Adjoint variables: $\lambda = (\lambda_{-1},\ \lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5)$

The Hamiltonian:

$$\begin{aligned}\mathcal{H} \quad = \quad & \lambda_{-1} - \frac{1}{2}\lambda_0(h_1\rho_1 + h_2\rho_2) + \frac{1}{2}\lambda_1(h_1\rho_0 + h_2\rho_3) \\ & + \frac{1}{2}\lambda_2(-h_1\rho_3 + h_2\rho_0) + \frac{1}{2}\lambda_3(h_1\rho_2 - h_2\rho_1) \\ & + \frac{1}{2}\lambda_4 h_1 + \frac{1}{2}\lambda_5 h_2\end{aligned}$$

Adjoint Equations
(Arthurs and Walsh, 1986)

Vector form: $\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial \rho}$

Coordinate form:

$$\dot{\lambda}_{-1} = 0, \qquad \dot{\lambda}_4 = 0, \qquad \dot{\lambda}_5 = 0$$

$$\dot{\lambda}_0 = -\frac{1}{2}(h_1\lambda_1 + h_2\lambda_2), \qquad \dot{\lambda}_1 = \frac{1}{2}(h_1\lambda_0 + h_2\lambda_3)$$

$$\dot{\lambda}_2 = \frac{1}{2}(-h_1\lambda_3 + h_2\lambda_0), \qquad \dot{\lambda}_3 = \frac{1}{2}(h_1\lambda_2 - h_2\lambda_1)$$

Rolling a Sphere Along a Curve of the Shortest Length
(a "minimum-time" optimal control problem)

Given the initial position $x(0) = z(0) = 0$ and the final position

$$x(T) \equiv \int_0^T h_2(t)\,\mathrm{d}t = x_E, \qquad z(T) \equiv \int_0^T h_1(t)\,\mathrm{d}t = -z_E$$

the initial and final orientation of the sphere

$$q(0) = (1, 0, 0, 0), \qquad q(T) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$$

find the optimal control $(h_1^\star, h_2^\star)$ (the optimal unit vector $\mathbf{h}^\star$ along the axis of rotation) such that minimizes the cost function $\mathcal{J} \equiv T$ (minimizes the length T of the curve).

Optimal Controls
(applying Pontryagin maximum principle)

The optimal controls $(h_1^\star, h_2^\star)$ are found by choosing

$$\lambda_{-1} = -1$$

(as for "minimum-time" problem) and maximizing the Hamiltonian $\mathcal{H}$ amongst the admissible controls

$$h_1^2 + h_2^2 - 1 = 0$$

The optimal controls are found to be (Hammersley, 1983)

$$\begin{aligned} h_1^\star &= \lambda_4 - \lambda_0\rho_1 + \lambda_1\rho_0 - \lambda_2\rho_3 + \lambda_3\rho_2 \\ h_2^\star &= \lambda_5 - \lambda_0\rho_2 + \lambda_1\rho_3 + \lambda_2\rho_0 - \lambda_3\rho_1 \end{aligned}$$

Curvature Equation I
(Arthurs and Walsh, 1986)

The coordinates of **h** are given by

$$h_1 = \cos\psi, \qquad h_1 = \sin\psi \tag{1}$$

where ψ is the angle between **h** and the coordinate axis Ox.

The curvature κ of a plane curve Γ is given by

$$\kappa = \dot{\psi} \tag{2}$$

Based on the state and adjoint equations $(\dot{\lambda} = -\frac{\partial\mathcal{H}}{\partial\rho},\ \dot{\rho} = \frac{\partial\mathcal{H}}{\partial\lambda})$, and the expression for $(h_1^\star, h_2^\star)$, it is deduced from (1) and (2) that

$$\dot{\kappa} = \lambda\cos(\psi + \varepsilon), \qquad \lambda, \varepsilon = \text{const}$$

Curvature Equation II
(Arthurs and Walsh, 1986)

On integrating the last equation

$$\dot{\kappa} = \lambda \cos(\psi + \varepsilon), \qquad \lambda > 0, \ \varepsilon = \text{const}$$

and making use of

$$\dot{x} = \sin(\psi + \varepsilon), \quad \dot{z} = -\cos(\psi + \varepsilon), \quad \varepsilon = \text{const}$$

it is obtained that the curvature of Γ satisfies the equation

$$\kappa = -\lambda z - \mu, \qquad \lambda > 0, \ \mu = \text{const}$$

which is equivalently written as

$$\frac{z''}{(1 + z'^2)^{3/2}} = -\lambda z - \mu \qquad (z' = dz/dx)$$

Curvature Equation III

On integrating the last equation

$$\frac{z''}{(1+z'^2)^{3/2}} = -\lambda z - \mu \qquad (z' = \mathrm{d}z/\mathrm{d}x)$$

under the initial condition

$$z' = 0 \quad \text{for} \quad z = \eta = \text{maximum deflection ordinate}$$

it is obtained the equation

$$(z')^2 = \frac{1 - \left[1 + \frac{\lambda}{2}(z^2 - \eta^2) + \mu(z - \eta)\right]^2}{\left[1 + \frac{\lambda}{2}(z^2 - \eta^2) + \mu(z - \eta)\right]^2}, \qquad \lambda > 0,\ \mu,\ \eta = \text{const}$$

Curvature Equation IV
The Intrinsic Equation of the Trajectory

On introducing the arc length parameter t, it follows from

$$\kappa = -\lambda z - \mu$$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 = \frac{1 - \left[1 + \frac{\lambda}{2}(z^2 - \eta^2) + \mu(z - \eta)\right]^2}{\left[1 + \frac{\lambda}{2}(z^2 - \eta^2) + \mu(z - \eta)\right]^2}$$

the intrinsic equation of the trajectory

$$\frac{\mathrm{d}\kappa}{\mathrm{d}t} = \frac{1}{4}(\sigma^2 - \kappa^2)(\kappa^2 + 4\lambda - \sigma^2), \qquad \sigma = \mu + \eta\lambda$$

The Plate-Ball Problem and the Eulerian Elastica

On substituting with

$$z = \zeta - \frac{\mu}{\lambda}, \qquad \sigma^2 = 2(1-\nu)\lambda$$

where ζ is a new coordinate and $\nu < 1$ is a new parameter, the equations of the plate-ball problem are reduced to the equations of the Eulerian elastica (Djondjorov, Hadzhilazova, Mladenov and Vassilev, 2008)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\lambda\zeta^2}{2} + \nu$$

$$\left(\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)^2 = -\frac{\lambda^2\zeta^4}{4} - \lambda\nu\zeta^2 - \nu^2 + 1$$

Trajectories of the Plate-Ball Problem

$\nu \in (-1, 1)$

$$x(t) = \frac{2}{\sqrt{\lambda}} E(\mathrm{am}(\sqrt{\lambda} t, k), k) - t, \qquad z(t) = a\,\mathrm{cn}(\sqrt{\lambda} t, k) - \frac{\mu}{\lambda}$$

where

$$a = \sqrt{\frac{2(1-\nu)}{\lambda}}, \qquad k = \sqrt{\frac{1-\nu}{2}}, \qquad \nu = 1 - \frac{\sigma^2}{2\lambda}$$

$$\sigma = \mu + \eta\lambda$$

$E(u, k)$ incomplete elliptic integral of second order
$\mathrm{am}(u, k)$ Jacobian amplitude function
$\mathrm{cn}(u, k)$ Jacobian elliptic cosine function

Trajectories of the Plate-Ball Problem

$\nu = -1$

$$x(t) = \frac{4\tanh(\sqrt{\lambda}t)}{\sqrt{\lambda}} - t, \qquad z(t) = \frac{4\,\mathrm{sech}(\sqrt{\lambda}t)}{\sqrt{\lambda}} - \frac{\mu}{\lambda}$$

$$\nu = 1 - \frac{\sigma^2}{2\lambda}, \qquad \sigma = \mu + \eta\lambda$$

Trajectories of the Plate-Ball Problem

$\nu < -1$

$$x(t) = aE(\mathrm{am}(\sqrt{\frac{\lambda(1-\nu)}{2}}t, k), k) + \nu t$$

$$z(t) = a\,\mathrm{dn}(\sqrt{\frac{\lambda(1-\nu)}{2}}t, k) - \frac{\mu}{\lambda}$$

where

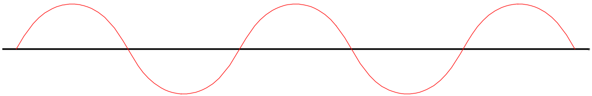
$$a = \sqrt{\frac{2(1-\nu)}{\lambda}}, \quad k = \sqrt{\frac{2}{1-\nu}}, \quad \nu = 1 - \frac{\sigma^2}{2\lambda}, \quad \sigma = \mu + \eta\lambda$$

$E(u, k)$ incomplete elliptic integral of second order
$\mathrm{am}(u, k)$ Jacobian amplitude function
$\mathrm{dn}(u, k)$ Jacobian elliptic delta function

Case I

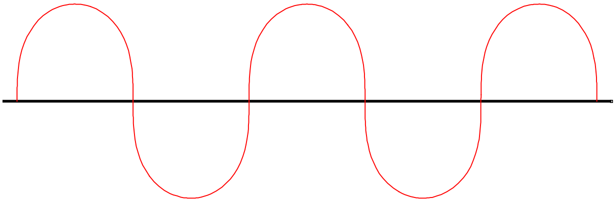
A Trajectory of the Plate-Ball Problem

(Arc of the Eulerian Elastica for $\lambda = 4$, $\sigma = 2$)



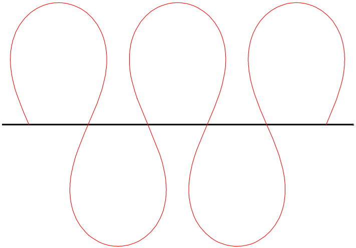
Case II

A Trajectory of the Plate-Ball Problem
(Arc of the Eulerian Elastica for $\lambda = 4$, $\sigma = 2.83$)



Case III

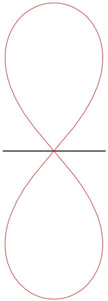
A Trajectory of the Plate-Ball Problem
(Arc of the Eulerian Elastica for $\lambda = 4$, $\sigma = 3.35$)



Case IV

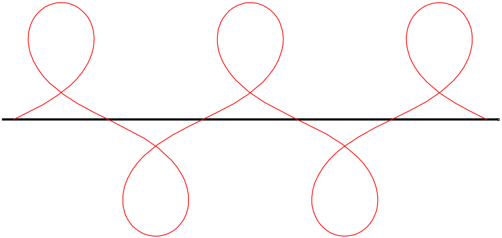
A Trajectory of the Plate-Ball Problem

(Arc of the Eulerian Elastica for $\lambda = 4$, $\sigma = 3.63564$)



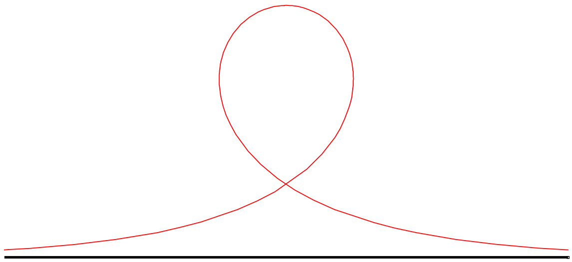
Case V

A Trajectory of the Plate-Ball Problem
(Arc of the Eulerian Elastica for $\lambda = 4$, $\sigma = 3.9$)



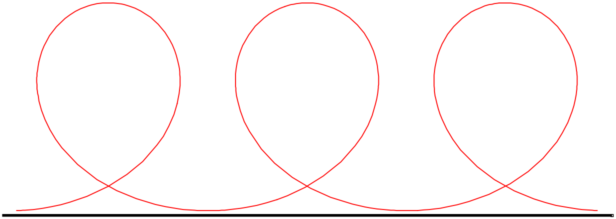
Case VI

A Trajectory of the Plate-Ball Problem
(Arc of the Eulerian Elastica for $\lambda = 4$, $\sigma = 4$)



Case VII

A Trajectory of the Plate-Ball Problem
(Arc of the Eulerian Elastica for $\lambda = 4$, $\sigma = 4.2$)



References

- Hammersley J. (2008) *Oxford Commemoration Ball*, In Probability, Statistics and Analysis, London Math. Soc. Lecture Note Series, Cambridge University Press, **79**, 112-142.
- Arthurs A. and Walsh G. (1986) *On Hammersley's Minimum Problem for a Rolling Sphere*, Math. Proc. Camb. Phil. Soc., 529-534.
- Jurdjevic V.(1993) *The geometry of the plate-ball problem*, Arch. Rat. Mech. Anal. **124** 305-328.
- Djondjorov P., Hadzhilazova M., Mladenov I. and Vassilev V. (2008) *Explicit Parametrization of Euler's Elastica*, Geometry Integrability & Quantization. **9** 175-186.
- Mladenov I. and Hadzhilazova M. (2013) *The Many Faces of Elastica* (in Bulgarian), Avangard Prima, Sofia.
- Abramowitz M. and Stegun I. (1972) *Handbook of Mathematical Functions*, New York, Dover.