On auto and hetero Bäcklund transformations for the Hénon-Heiles systems

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Introduction

We consider a canonical transformation of parabolic coordinates on the plain associated with integrable Hénon-Heiles systems and suppose that this transformation together with some additional relations may be considered as a counterpart of the auto and hetero Bäcklund transformations.

Bäcklund transformations

Bäcklund transformation between two given PDEs

$$E_1(u,x,t)=0 \quad \text{and} \quad E_2(v,y,\tau)=0$$

is a pair of relations

$$F_{1,2}(u,x,t,v,y,\tau)=0 \tag{0.1}$$

and some additional relations between (x,t) and (y,τ), which allow to get both equations $E_{1,2}$.

Auto and hetero BT

Auto Bäcklund transformations

A counterpart of the auto-BTs — canonical transformation

$$(u, p_u) \to (v, p_v)\,, \qquad \{u_i, p_{u_j}\} = \{v_i, p_{v_j}\} = \delta_{ij}\,, \qquad i,j = 1,\dots,n, \tag{0.2}$$

preserving the algebraic form of the Hamilton-Jacobi equations

$$H_i\left(u, \frac{\partial S}{\partial u}\right) = \alpha_i \qquad \text{and} \qquad H_i\left(v, \frac{\partial S}{\partial v}\right) = \alpha_i$$

associated with the Hamiltonians $H_1, \dots, H_n$.

Auto and hetero BT

Hetero Bäcklund transformations

A counterpart of the hetero-BTs has to be a canonical transformation, which has to relate two different systems of the Hamilton-Jacobi equations

$$H_i\left(u, \frac{\partial S}{\partial u}\right) = \alpha_i \qquad \text{and} \qquad \tilde{H}_i\left(v, \frac{\partial \tilde{S}}{\partial v}\right) = \tilde{\alpha}_i \tag{0.3}$$

and has to satisfy some additional conditions.

Auto and hetero BT

One of the possible conditions:
Let us consider integrals of motion for the two-dimensional harmonic oscillator

- $H_1 = p_x^2 + p_y^2 + a(x^2 + y^2)\,, \qquad H_2 = p_x^2 - p_y^2 + a(x^2 - y^2)$
- $\tilde{H}_1 = p_x^2 + p_y^2 + a(x^2 + y^2)\,, \qquad \tilde{H}_2 = xp_y - yp_x$

Auto and hetero BT

Canonical transformation of variables

$$(u, p_u) = (x, y, p_x, p_y) \to (v, p_v) = (r, \varphi, p_r, p_\varphi) \tag{0.4}$$

defines a correspondence between the two different systems

$$H_{1,2}\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) = \alpha_{1,2} \qquad \text{and} \qquad \tilde{H}_{1,2}\left(r, \varphi, \frac{\partial \tilde{S}}{\partial r}, \frac{\partial \tilde{S}}{\partial \varphi}\right) = \tilde{\alpha}_{1,2}\,.$$

Auto and hetero BT

This correspondence may be considered as a hetero-BT defined by:

- generating function: $F = p_x r \cos\varphi + p_y r \sin\varphi$
- relations between (x, y) and (r, φ)
- $H_1 = \tilde{H}_1$ is *simultaneously separable* in u and v variables

The main aim:
Discuss a correspondence between integrable Hénon-Heiles systems which may be considered as a counterpart of the generic hetero-BTs relating different but *simultaneously separable* in v-variables Hamilton-Jacobi equations.

Integrable systems

Hamilton function on $T^*\mathbb{R}^n$:

$$H = p_1^2 + p_2^2 + V(q_1, q_2)\,. \tag{0.5}$$

Additively separated complete integral:

$$S(u_1, \ldots, u_n; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} S_i(u_i; \alpha_1, \ldots, \alpha_n)\,,$$

Momenta:

$$p_{u_i} = \frac{\partial S_i(u_i; \alpha_1, \ldots, \alpha_n)}{\partial u_i}\,, \qquad i = 1, \ldots, n.$$

Some Hamiltonian on the plain separable in parabolic coordinates $u_{1,2}$:

$$H = p_1^2 + p_2^2 + V(q_1, q_2)\,. \tag{0.6}$$

Integrable perturbation of H

$$\tilde{H}_1 = \frac{1}{2}\left(p_{u_1}^2 + U_1(u_1) + p_{u_1}^2 + U_2(u_2)\right) = H_1 + \frac{H_2}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right)$$

Second independent integral of motion

$$\tilde{H}_2 = \left(p_{u_1}^2 + U_1(u_1) - p_{u_1}^2 - U_2(u_2)\right) = H_2\left(\frac{1}{u_1} - \frac{1}{u_2}\right)$$

Hénon-Heiles systems

There are three integrable Hénon-Heiles systems on the plane, which can be identified with appropriate finite-dimensional reductions of the integrable fifth order KdV, Kaup-Kupershmidt and Sawada-Kotera equations.

Hénon-Heiles systems

- Lax matrix for the first Hénon-Heiles system

$$L(\lambda)=\begin{pmatrix} \frac{p_2}{2}+\frac{p_1q_1+\mathrm{i}b_1}{\lambda} & \lambda-2q_2-\frac{q_1^2}{\lambda} \\ a\lambda^2+2aq_2\lambda+a(q_1^2+4q_2^2)+\frac{p_1^2+b_1^2q_2^{-2}}{4\lambda} & -\frac{p_2}{2}-\frac{p_1q_1-\mathrm{i}b_1}{\lambda} \end{pmatrix}$$

- Characteristic polynomial: $det\left(L\left(\lambda\right)-\mu\right)=\mu^2-a\lambda^3-\frac{H_1}{4}+\frac{H_2}{\lambda}$
- Hamiltonian function of the first Hénon-Heiles system $H_1=p_1^2+p_2^2-16aq_2(q_1^2+2q_2^2)$
- Second integral of motion $H_2=aq_1^2(q_1^2+4q_2^2)+\frac{p_1(q_2p_1-q_1p_2)}{2}$

$$\Downarrow$$

$$\hat{L}(\lambda)=VLV^{-1}$$

$$V = \begin{pmatrix} L_{12} & 0 \\ 4\big(L_{11} - \hat{L}_{11}(\lambda)\big) & 4L_{12} \end{pmatrix}$$

$\hat{L}(\lambda)$ is defined by the following conditions:

1. $\hat{L}_{12}(\lambda) = \dfrac{(\lambda - u_1)(\lambda - u_2)}{4\lambda}$
2. $\hat{L}_{21} = 4a(\lambda - v_1)(\lambda - v_2)$
3. the conjugated momenta for u and v variables are the values of the diagonal element

$$p_{u_i} = \hat{L}_{11}(\lambda = u_i)\,, \qquad p_{v_i} = \hat{L}_{11}(\lambda = v_i)\,, \qquad i = 1, 2.$$

$$\hat{L}_{11}(\lambda) = \frac{p_2}{2} + \frac{p_1(\lambda - 2q_2)}{2q_1}$$

Hénon-Heiles systems

Entries of $\hat{L}$ in (u, p_u) and (v, p_v):

$$\hat{L}_{11} = \frac{\lambda - u_2}{u_1 - u_2} p_{u_1} + \frac{\lambda - u_1}{u_2 - u_1} p_{u_2} = \frac{\lambda - v_2}{v_1 - v_2} p_{v_1} + \frac{\lambda - v_1}{v_2 - v_1} p_{v_2}$$

$$\begin{aligned} \hat{L}_{12} &= \frac{(\lambda - u_1)(\lambda - u_2)}{4\lambda} \\ &= \frac{\lambda^2 + \lambda(v_1 + v_2) + v_1^2 + v_1 v_2 + v_2^2}{4\lambda} - \frac{(p_{v_1} - p_{v_2})^2}{4a(v_1 - v_2)^2} - \frac{p_{v_1}^2 - p_{v_2}^2}{4\lambda(v_1 - v_2)} \end{aligned}$$

$$\hat{L}_{21} = 4a(\lambda - v_1)(\lambda - v_2)$$

$$
\begin{aligned}
u_{1,2} &= -\frac{v_1+v_2}{2} + \frac{(p_{v_1}-p_{v_2})^2 \pm \sqrt{A}}{2a(v_1-v_2)^2},\\
p_{u_{1,2}} &= \frac{(p_{v_1}-p_{v_2})\left((p_{v_1}-p_{v_2})^2 \pm \sqrt{A}\right)}{2a(v_1-v_2)^3} - \frac{p_{v_1}(v_1+3v_2)-p_{v_2}(v_2+3v_1)}{2(v_1-v_2)}
\end{aligned}
$$

where

$$
\begin{aligned}
A &= (p_{v_1}-p_{v_2})^4 + 2a(v_1-v_2)^2(p_{v_1}-p_{v_2})\left(p_{v_1}(v_1-3v_2)-p_{v_2}(v_2-3v_1)\right)\\
&- a^2(3v_1^2+2v_1v_2+3v_2^2)(v_1-v_2)^4\,.
\end{aligned}
$$

Proposition

The auto-BT for the first Hénon-Heiles system is a correspondence between two equivalent systems of the Hamilton-Jacobi equations

$$H_{1,2}\left(\lambda, \frac{\partial S}{\partial \lambda}\right) = \alpha_{1,2}\,, \qquad \lambda = u, v\,,$$

where variables (u, p_u) *and* (v, p_v) *are related by canonical transformation and Hamiltonians* $H_{1,2}$ *are defined by the following equations*

$$\Phi(\lambda, \mu) = \mu^2 - a\lambda^3 = \frac{H_1}{4} - \frac{H_2}{\lambda}\,, \qquad \lambda = u_{1,2}, v_{1,2}, \quad \mu = p_{u_{1,2}}, p_{v_{1,2}}\,. \tag{0.7}$$

Hénon-Heiles systems

Proposition

For the three Hénon-Heiles systems on the plane an analogue of the hetero-BT is the correspondence between two different systems of the Hamilton-Jacobi equations

$$H_{1,2}\left(u,\frac{\partial S}{\partial u}\right)=\alpha_{1,2}\qquad and \qquad \tilde{H}_{1,2}\left(v,\frac{\partial \tilde{S}}{\partial v}\right)=\tilde{\alpha}_{1,2}\,,$$

where variables (u,p_u) and (v,p_v) are related by canonical transformation and Hamiltonians are defined by the following equations

$$\Phi(\lambda,\mu)=\mu^2-a\lambda^3=\frac{H_1}{4}-\frac{H_2}{\lambda}\,,\qquad \lambda=u_{1,2},v_{1,2},\quad \mu=p_{u_{1,2}},p_{v_{1,2}}\,,$$

and

$$\tilde{H}_{1,2}=\Phi(v_1,p_{v_1})\pm\Phi(v_2,p_{v_2})\,.$$

Hamilton function:

$$H = p_1^2 + p_2^2 + V_N(q_1, q_2)\,, \qquad V_N = 4a \sum_{k=0}^{[N/2]} 2^{1-2k} \begin{pmatrix} N-k \\ k \end{pmatrix} q_1^{2k} q_2^{N-2k}\,,$$

At $N = 4$ — "(1:12:16)" system with Hamiltonian

$$H = p_1^2 + p_2^2 - 4a\left(q_1^4 + 12q_1^2q_2^2 + 16q_2^4\right)\,. \qquad (0.8)$$

Integrable Hamiltonian with velocity dependent potential

Lax matrix:

$$L(\lambda) = \begin{pmatrix} \frac{p_2}{2} + \frac{p_1 q_1}{2\lambda} & \lambda - 2q_2 - \frac{q_1^2}{\lambda} \\ a\lambda^3 + 2aq_2\lambda^2 + a(q_1^2 + 4q_2^2)\lambda + 4aq_2(q_1^2 + 2q_2^2) + \frac{p_1^2}{4\lambda} & -\frac{p_2}{2} - \frac{p_1 q_1}{2\lambda} \end{pmatrix}.$$

After similarity transformation of $L(\lambda)$ with matrix V where

$$\hat{L}_{11}(\lambda) = \sqrt{a}\,\lambda^2 - \frac{4\sqrt{a}q_2 q_1 - p_1}{2q_1}\,\lambda - \frac{2\sqrt{a}q_1^3 + 2p_1 q_2 - p_2 q_1}{2q_1}$$

Integrable Hamiltonian with velocity dependent potential

- First coordinates are parabolic coordinates $u_{1,2}$
- Second coordinates $v_{1,2}$ are zeroes of the polynomial:

$$\hat{L}_{21} = \frac{4(4aq_1q_2 - \sqrt{a}p_1)}{q_1}(\lambda - v_1)(\lambda - v_2)$$

Hamiltonian with velocity dependent potential:

$$\tilde{H} = p_1^2 + p_2^2 - 3\sqrt{a}p_2q_1^2 + 2a(q_1^4 - 12q_1^2q_2^2 - 32q_2^4) \qquad (0.9)$$

Integrable Hamiltonian with velocity dependent potential

Fourth order polynomial in momenta:

$$
\begin{aligned}
\tilde{H}_2 &= p_1^4 + 4q_1^4\left(q_1^4 - 8q_1^2q_2^2 - 112q_2^4\right)a^2 \\
&+ 4q_1^3\left(64p_1q_2^3 - p_2q_1^3 - 12p_2q_1q_2^2\right)a^{3/2} \\
&+ q_1^2\left(4p_1^2q_1^2 - 48p_1^2q_2^2 + 32p_1p_2q_1q_2 + p_2^2q_1^2\right)a - 6a^{1/2}p_1^2p_2q_1^2\,,
\end{aligned}
$$

Conclusion

We have constructed a canonical transformation of the standard parabolic coordinates, which yields variables of separation for the three integrable Hénon-Heiles systems.
Moreover, we believe that information about such suitable Bäcklund transformations and the corresponding integrable systems is incorporated into Lax matrices associated with these elliptic, parabolic etc. coordinates.
In order to prove it we obtained integrals of motion, variables of separation and separated relations for some new integrable system with velocity dependent potential and fourth order integral of motion in momenta.