Centro-Affine hypersurfaces with an induced almost paracontact structure

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03-08.06.2016 Varna

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Previous results

In the paper **„$\widetilde{J}$-tangent affine hypersurfaces with an induced almost paracontact structure"** (submitted) I studied affine hypersurfaces $f\colon M \to \mathbb{R}^{2n+2}$ with an arbitrary $\widetilde{J}$-tangent transversal vector field, where $\widetilde{J}$ is the canonical paracomplex structure on $\mathbb{R}^{2n+2}$. Such a vector field induces in a natural way an almost paracontact structure (φ, ξ, η) as well as the second fundamental form h. It was proved that if (φ, ξ, η, h) is an almost paracontact metric structure then it is a para α-Sasakian structure with $\alpha = -1$. Moreover, the hypersurface must be a piece of a hyperquadric.

Affine immersions

Let $f\colon M \to \mathbb{R}^{n+1}$ be an orientable connected differentiable n-dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then for any transversal vector field C we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X,Y)C \qquad \text{(Gauss' formula)}$$

and

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \qquad \text{(Weingarten's formula)}$$

where X, Y are vector fields tangent to M. Here

- ∇ — torsion free connection called *the induced connection*,
- h — tensor of type (0,2) called *the second fundamental form*,
- S — tensor of type (1,1) called *the shape operator*,
- τ — 1-form called *the transversal connection form*.

Affine immersions

We have the following

Fundamental equations, [Nomizu, Sasaki]

For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY, \tag{1}$$

$$(\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) = (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \tag{2}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{3}$$

$$h(X,SY) - h(SX,Y) = 2d\tau(X,Y). \tag{4}$$

Centro-affine hypersurface

Let o be a point of the affine space $\mathbb{R}^{n+1}$ chosen as origin. An immersion f of an n-manifold M into $\mathbb{R}^{n+1} \setminus \{o\}$ such that $C = -\overrightarrow{of(x)}$ for every $x \in M$ is always transversal to $f_* TM$ is called *centro-affine hypersurface*.

Blaschke hypersurface

We say that f is *nondegenerate* if the second fundamental form h is nondegenerate.
For a nondegenerate (orientable) hypersurface there exists a (global) transversal vector field C satisfying the conditions:

$$\nabla\theta = 0, \qquad \theta = \omega_h,$$

where ω_h is a volume element determined by h

$$\omega_h(X_1, \ldots, X_n) := \sqrt{|det[h(X_i, X_j)]_{i,j=1\ldots n}|}$$

and θ is an induced volume element on M

$$\theta(X_1, \ldots, X_n) := det[f_* X_1, \ldots, f_* X_n, C].$$

A transversal vector field satisfying these conditions is called *the affine normal field* or *the Blaschke normal field*. It is unique up to sign. A hypersurface with the transversal Blaschke normal field is called *the Blaschke hypersurface*.

Affine hyperspheres

A Blaschke hypersurface is called *an affine hypersphere* if $S = \lambda I$, where $\lambda = const$.
If $\lambda = 0$, f is called *an improper affine hypersphere*, if $\lambda \neq 0$, f is called *a proper affine hypersphere.*

Affine hypeersurfaces with a $\widetilde{J}$-tangent transversal vector field

From now on we are interested in $(2n+1)$-dimensional hypersurfaces $f\colon M \mapsto \mathbb{R}^{2n+2}$. We assume that $\mathbb{R}^{2n+2}$ is endowed with the standard paracomplex structure $\widetilde{J}$, that is

$$\widetilde{J}(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_{n+1}, x_1, \ldots, x_{n+1}).$$

Definition 1.

A transversal vector field C will be called *$\widetilde{J}$-tangent*, if $\widetilde{J}C \in f_*(TM)$.

The biggest $\widetilde{J}$ invariant distribution on M we denote by $\mathcal{D}$. That is

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap \widetilde{J}(f_*(T_xM)))$$

for every $x \in M$. We have that $\dim \mathcal{D}_x \geq 2n$. If for some x the $\dim \mathcal{D}_x = 2n+1$ then $\mathcal{D}_x = T_xM$ and it is not possible to find $\widetilde{J}$-tangent transversal vector field in a neighbourhood of x. Since we study only hypersurfaces with a $\widetilde{J}$-tangent transversal vector field we always have $\dim \mathcal{D} = 2n$. The distribution $\mathcal{D}$ is smooth, since $\dim \mathcal{D}$ is constant and is an intersection of two smooth distributions.
A vector field X is called a *$\mathcal{D}$-field* if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields.

Almost paracontact structures

A $(2n+1)$-dimensional manifold M is said to have an *almost paracontact structure* if there exist on M a tensor field φ of type (1,1), a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = X - \eta(X)\xi, \tag{5}$$

$$\eta(\xi) = 1 \tag{6}$$

for every $X \in TM$ and the tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker\eta$, that is the eigendistributions $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigenvalues $1, -1$ of φ have equal dimension n.

Induced almost paracontact structure

Definition 2.

Let $f\colon M \to \mathbb{R}^{2n+2}$ ($\dim M = 2n+1$) be a hypersurface with a $\widetilde{J}$-tangent transversal vector field C. Then we define a vector field ξ, a 1-form η and a tensor field φ of type (1,1) as follows:

$$\xi := \widetilde{J}C,$$

$$\eta|_{\mathcal{D}} = 0 \text{ and } \eta(\xi) = 1,$$

$$\varphi|_{\mathcal{D}} = \widetilde{J}|_{\mathcal{D}} \text{ and } \varphi(\xi) = 0.$$

A structure (φ, ξ, η) is called *an induced almost paracontact structure on* M.

Induced almost paracontact structure

Theorem 1.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with $\widetilde{J}$-tangent transversal vector field C. If (φ, ξ, η) is an induced almost paracontact structure on M then the following equations hold:

$$\eta(\nabla_X Y) = h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{7}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y - \eta(Y)SX - h(X, Y)\xi, \tag{8}$$

$$\begin{aligned}\eta([X, Y]) = h(X, \varphi Y) - h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \\ + \eta(Y)\tau(X) - \eta(X)\tau(Y),\end{aligned} \tag{9}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \eta(X)SY - \eta(Y)SX, \tag{10}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{11}$$

$$\eta(SX) = -h(X, \xi) \tag{12}$$

for every $X, Y \in \mathcal{X}(M)$.

Proof. For every $X \in TM$ we have

$$\widetilde{J}X = \varphi X + \eta(X)C.$$

Furthermore

$$\begin{aligned}\widetilde{J}(D_X Y) &= \widetilde{J}(\nabla_X Y + h(X,Y)C) = \widetilde{J}(\nabla_X Y) + h(X,Y)\widetilde{J}C \\ &= \varphi(\nabla_X Y) + \eta(\nabla_X Y)C + h(X,Y)\xi\end{aligned}$$

and

$$\begin{aligned}D_X \widetilde{J}Y &= D_X(\varphi Y + \eta(Y)C) = D_X \varphi Y + X(\eta(Y))C + \eta(Y)D_X C \\ &= \nabla_X \varphi Y + h(X,\varphi Y)C + X(\eta(Y))C + \eta(Y)(-SX + \tau(X)C) \\ &= \nabla_X \varphi Y - \eta(Y)SX + (h(X,\varphi Y) + X(\eta(Y) + \eta(Y)\tau(X))C.\end{aligned}$$

Since $\mathrm{D}_X \widetilde{J}Y = \widetilde{J}(D_X Y)$, comparing transversal and tangent parts, we obtain (7) and (8) respectively. Equations (9)—(12) follow directly from (7) and (8).

Affine hypersurface of codimension two

Let $f\colon M \to \mathbb{R}^{n+2}$ be an immersion, and $\mathcal{N}\colon M \ni x \mapsto N_x$ be a transversal bundle for the immersion f. Immersion f together with the transversal bundle $\mathcal{N}$ we call an *affine hypersurface of codimension two*

Para-holomorfic hypersurface

Let $g \colon M^{2n} \to \mathbb{R}^{2n+2}$ be an immersion and let $\widetilde{J}$ be the standard paracomplex structure on $\mathbb{R}^{2n+2}$. We always identify $(\mathbb{R}^{2n+2}, \widetilde{J})$ with $\widetilde{\mathbb{C}}^{n+1}$, where $\widetilde{\mathbb{C}}$ is the real algebra of para-complex numbers.
We assume that $g_*(TM)$ is $\widetilde{J}$-invariant and $\widetilde{J}|_{g_*(T_xM)}$ is a para-complex structure on $g_*(T_xM)$ for every $x \in M$. Then $\widetilde{J}$ induces an almost para-complex structure on M which we will also denote by $\widetilde{J}$. Moreover, since $(\mathbb{R}^{2n+2}, \widetilde{J})$ is para-complex then $(M, \widetilde{J})$ is para-complex as well. By assumption we have that $dg \circ \widetilde{J} = \widetilde{J} \circ dg$ that is $g \colon M^{2n} \to \mathbb{R}^{2n+2} \cong \widetilde{\mathbb{C}}^{n+1}$ is a para-holomorphic immersion.
Since para-complex dimension of M is n, immersion g is called a *para-holomorphic hypersurface*.

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be an affine hypersurface of codimension 2 with a transversal bundle $\mathcal{N}$.
If g is para-holomorphic then it is called *affine para-holomorphic hypersurface*. If additionally the transversal bundle $\mathcal{N}$ is $\widetilde{J}$-invariant then g is called a *para-complex affine hypersurface*.

Definition 3.

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. We say that g is *para-complex centro-affine hypersurface* if $\{g, \widetilde{J}g\}$ is a transversal bundle for g.

Theorem 2.

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. Then for every $x \in M$ there exists a neighborhood U of x and a transversal vector field $\zeta\colon U \to \mathbb{R}^{2n+2}$ such that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for $g|_U$. That is $g|_U$ considered with $\{\zeta, \widetilde{J}\zeta\}$ is a para-complex affine hypersurface.

Proof. Indeed, let N_x be any vector space transversal to $g_*(T_xM)$. If N_x is $\widetilde{J}$-invariant then it must be a para-complex vector space so we can find vector $v \in N_x$ such that $\{v, \widetilde{J}v\}$ is a basis for N_x. If N_x is not $\widetilde{J}$-invariant then $N_x \cap \widetilde{J}N_x$ must be 1-dimensional. In this case we can choose $v \in N_x$ such that $v \notin N_x \cap \widetilde{J}N_x$. Now vector $\widetilde{J}v$ is transversal to $g_*(T_xM)$ and linearly independent with v. That is $\{v, \widetilde{J}v\}$ is a para-complex transversal vector space to $g_*(T_xM)$.
Summarizing at x we can always find a transversal vector v such that $g_*(T_xM) \oplus \operatorname{span}\{v, \widetilde{J}v\} = \mathbb{R}^{2n+2}$.
Hence in a neighborhood of x we can find a transversal vector field ζ such that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g in this neighborhood.

Now, let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface and let $\zeta\colon U \to \mathbb{R}^{2n+2}$ be a local transversal vector field on $U \subset M$ such that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle to g.
So for all tangent vector fields $X, Y \in \mathcal{X}(U)$ we can decompose $D_X Y$ and $D_X \zeta$ into tangent and transversal part.
That is we have

$$\mathrm{D}_X\, g_* Y = g_*(\nabla_X Y) + h_1(X,Y)\zeta + h_2(X,Y)\widetilde{J}\zeta \quad \text{(formula of Gauss)}$$

$$\mathrm{D}_X\, \zeta = -g_*(SX) + \tau_1(X)\zeta + \tau_2(X)\widetilde{J}\zeta \quad \text{(formula of Weingarten)}$$

where ∇ is a torsion free affine connection on U, h_1 and h_2 are symmetric bilinear forms on U, S is a $(1,1)$-tensor field on U and τ_1 and τ_2 are 1-forms on U.

Using the fact that $D\widetilde{J} = 0$ and the formula of Gauss by straightforward computations we can prove the following

Lemma 1.

$$\nabla \widetilde{J} = 0, \tag{13}$$

$$h_1(X, \widetilde{J}Y) = h_1(\widetilde{J}X, Y) = h_2(X, Y), \tag{14}$$

$$h_2(X, \widetilde{J}Y) = h_1(X, Y). \tag{15}$$

We say that a hypersurface is *nondegenerate* if h_1 (and in consequence h_2) is nondegenerate.

Now assume that $\{\widetilde{\zeta}, \widetilde{J\zeta}\}$ is any other transversal bundle on U. Then there exist functions φ, ψ on U and $Z \in \mathcal{X}(U)$ such that

$$\widetilde{\zeta} = \varphi\zeta + \psi \widetilde{J}\zeta + g_* Z.$$

Since $\{\widetilde{\zeta}, \widetilde{J\zeta}\}$ is transversal the above formula implies that $\varphi^2 - \psi^2 \neq 0$. Indeed, we have

$$\varphi\widetilde{\zeta} - \psi\widetilde{J\zeta} = (\varphi^2 - \psi^2)\zeta + \varphi g_* Z - \psi\widetilde{J} g_* Z.$$

If $\varphi^2 - \psi^2 = 0$ then $\varphi\widetilde{\zeta} - \psi\widetilde{J\zeta} \in TU$, but since $\{\widetilde{\zeta}, \widetilde{J\zeta}\}$ is transversal we obtain $\varphi = \psi = 0$ what is impossible because $\widetilde{\zeta}$ is transversal.

By the formulas of Gauss and Weingarten with respect to $\widetilde{\zeta}$ we obtain the objects $\widetilde{\nabla}$, $\widetilde{h}_1$, $\widetilde{h}_2$, $\widetilde{S}$, $\widetilde{\tau}_1$, $\widetilde{\tau}_2$ which satisfy the following relations

Lemma 2.

$$h_1(X, Y) = \varphi\widetilde{h}_1(X, Y) + \psi\widetilde{h}_2(X, Y), \tag{16}$$

$$h_2(X, Y) = \psi\widetilde{h}_1(X, Y) + \varphi\widetilde{h}_2(X, Y), \tag{17}$$

$$\nabla_X Y = \widetilde{\nabla}_X Y + \widetilde{h}_1(X, Y)Z + \widetilde{h}_2(X, Y)\widetilde{J}Z, \tag{18}$$

$$\varphi SX + \psi SX - \nabla_X Z = \widetilde{S}X - \widetilde{\tau}_1(X)Z - \widetilde{\tau}_2(X)\widetilde{J}Z, \tag{19}$$

$$\varphi\widetilde{\tau}_1(X) + \psi\widetilde{\tau}_2(X) = X(\varphi) + \varphi\tau_1(X) + \psi\tau_2(X) + h_1(X, Z), \tag{20}$$

$$\psi\widetilde{\tau}_1(X) + \varphi\widetilde{\tau}_2(X) = \varphi\tau_2(X) + X(\psi) + \psi\tau_1(X) + h_2(X, Z), \tag{21}$$

$$\widetilde{h}_1 = \frac{h_1\varphi - h_2\psi}{\varphi^2 - \psi^2}, \tag{22}$$

$$\widetilde{\tau}_1(X) = \frac{1}{2}X(\ln|\varphi^2 - \psi^2|) + \tau_1(X) + \frac{1}{\varphi^2 - \psi^2}(\varphi h_1(X, Z) - \psi h_2(X, Z)). \tag{23}$$

Proof. Formulas (16) to (21) are straightforward. Formulas (22) and (23) follow at once from (16), (17), (20) and (21).

On U we define the volume form θ_ζ by the formula

$$\theta_\zeta(X_1, \dots, X_{2n}) := \det(g_* X_1, \dots, g_* X_{2n}, \zeta, \widetilde{J}\zeta)$$

for tangent vectors X_i, $i = 1, \dots, 2n$. Then, consider the function H_ζ on U defined by

$$H_\zeta := \det[h_1(X_i, X_j)]_{i,j=1\dots 2n}$$

where $X_1, \dots, X_{2n}$ is a local basis in TU such that $\theta_\zeta(X_1, \dots, X_{2n}) = 1$. This definition is independent of the choice of basis.
Moreover, we also have

$$\nabla_X \theta_\zeta = 2\tau_1(X)\theta_\zeta.$$

If $\{\widetilde{\zeta}, \widetilde{J\zeta}\}$ is other transversal bundle on U then we have the following relations between $\theta_{\widetilde{\zeta}}$, $H_{\widetilde{\zeta}}$ and θ_ζ, H_ζ

Lemma 3.

$$\theta_{\widetilde{\zeta}} = (\varphi^2 - \psi^2)\theta_\zeta, \tag{24}$$

$$H_{\widetilde{\zeta}} = \frac{1}{(\varphi^2 - \psi^2)^{n+2}} \cdot H_\zeta. \tag{25}$$

Proof. Formula (24) is straightforward. So, we only prove (25). Let $\{X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n\}$ be a local basis on TM. Then

$$\theta_\zeta(X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n) = \alpha$$

where $\alpha \neq 0$ (so either $\alpha < 0$ or $\alpha > 0$). Now let $\widetilde{X_1} := \frac{X_1}{\sqrt{|\alpha|}}$ then

$$\theta_\zeta(\widetilde{X_1}, \widetilde{J}\widetilde{X_1}, X_2, \widetilde{J}X_2, \ldots, X_n, \widetilde{J}X_n) = \frac{\alpha}{|\alpha|}.$$

So we can choose the basis $\{X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n\}$ such that

$$\theta_\zeta(X_1, \widetilde{J}X_1, \ldots, X_n, \widetilde{J}X_n) = \pm 1.$$

Let $Y_i = \frac{X_i}{|\varphi^2-\psi^2|^{\frac{1}{2n}}}$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}\theta_{\widetilde{\zeta}}(Y_1, \ldots, \widetilde{J}Y_n) &= (\varphi^2 - \psi^2)\theta_{\zeta}(Y_1, \ldots, \widetilde{J}Y_n)\\ &= (\varphi^2 - \psi^2) \cdot \frac{1}{|\varphi^2 - \psi^2|}\theta_{\zeta}(X_1, \ldots, X_{2n})\\ &= sgn(\varphi^2 - \psi^2)\theta_{\zeta}(X_1, \ldots, X_{2n}) = \pm 1,\end{aligned}$$

so

$$\begin{aligned}H_{\widetilde{\zeta}} &= \det\left[\widetilde{h}_1(Y_i, Y_j)\right]\\ &= \frac{1}{(\varphi^2 - \psi^2)^2}\det\left[\widetilde{h}_1(X_i, X_j)\right].\end{aligned}$$

We also compute

$$\det\begin{bmatrix}\widetilde{h}_1(X_k, X_l) & \widetilde{h}_1(X_k, \widetilde{J}X_l)\\ \widetilde{h}_1(X_m, X_l) & \widetilde{h}_1(X_m, \widetilde{J}X_l)\end{bmatrix} = \frac{1}{\varphi^2-\psi^2}\det\begin{bmatrix}h_1(X_k, X_l) & h_1(X_k, \widetilde{J}X_l)\\ h_1(X_m, X_l) & h_1(X_m, \widetilde{J}X_l)\end{bmatrix}.$$

The above implies that

$$H_{\widetilde{\zeta}} = \frac{1}{(\varphi^2-\psi^2)^2}\cdot\frac{1}{(\varphi^2-\psi^2)^n}\cdot H_\zeta$$

and eventually

$$H_{\widetilde{\zeta}} = \frac{1}{(\varphi^2-\psi^2)^{n+2}}\cdot H_\zeta.$$

Affine normal vector fields

Definition 4.

When g is nondegenerate there exist transversal vector fields ζ satisfying the following two conditions:

$$|H_\zeta| = 1,$$
$$\tau_1 = 0.$$

Such vector fields are called *affine normal vector fields*.

Proof. Let $\{\zeta, \widetilde{J}\zeta\}$ be an arbitrary transversal bundle for g. Since g is nondegenerate we have $H_\zeta \neq 0$ so we can find functions φ and ψ such that $\varphi^2 - \psi^2 \neq 0$ and

$$|(\varphi^2 - \psi^2)^{n+2}| = |H_\zeta|. \tag{26}$$

Let $\widetilde{\zeta} := \varphi\zeta + \psi\zeta + Z$ where Z is an arbitrary vector field on M. Lemma 3 (formula (25)) and formula (26) imply that $|H_{\widetilde{\zeta}}| = 1$.

We shall show that we can choose Z in such a way that $\widetilde{\zeta}$ is an affine normal vector field.
By Lemma 2 (formula (23)) we have

$$\widetilde{\tau}_1(X) = \frac{1}{2}X(\ln|\varphi^2 - \psi^2|) + \tau_1(X) + \frac{1}{\varphi^2 - \psi^2}(\varphi h_1(X, Z) - \psi h_2(X, Z))$$

Now using Lemma 1 we obtain

$$\widetilde{\tau}_1(X) = \frac{1}{2}X(\ln|\varphi^2 - \psi^2|) + \tau_1(X) + \frac{1}{\varphi^2 - \psi^2} \cdot h_1(X, \varphi Z - \psi \widetilde{J}Z).$$

Since h_1 is nondegenerate we can find Z such that $\widetilde{\tau}_1(X) = 0$ for all vector fields X defined on U. In this way we have shown that on every para-holomorphic hypersurface one may find (at least locally) an affine normal vector field.

Lemma 4.

Let $g\colon M^{2n} \to \mathbb{R}^{2n+2}$ be a nondegenerate para-holomorphic hypersurface and let $\zeta, \widetilde{\zeta}\colon U \to \mathbb{R}^{2n+2}$ be two affine normal vector fields on $U \subset M$. Then $\widetilde{\zeta} = \varphi\zeta + \psi\widetilde{J}\zeta$ where $|\varphi^2 - \psi^2| = 1$.

Proof. Since $\zeta, \widetilde{\zeta}$ are transversal there exist functions $\varphi, \psi \in C^\infty(U)$ and a tangent vector field $Z \in \mathcal{X}(U)$ such that $\widetilde{\zeta} = \varphi\zeta + \psi\widetilde{J}\zeta + Z$. Since $|H_\zeta| = |H_{\widetilde{\zeta}}| = 1$ the formula (25) implies that $|\varphi^2 - \psi^2| = 1$. Now, due to the fact that $\tau_1 = \widetilde{\tau}_1 = 0$ and by formulas (23) and Lemma 1 we obtain

$$0 = \varphi h_1(X, Z) - \psi h_2(X, Z) = \varphi h_1(X, Z) - \psi h_1(X, \widetilde{J}Z) = h_1(X, \varphi Z - \psi\widetilde{J}Z)$$

for all $X \in \mathcal{X}(U)$. Since h_1 is non-degenerate and $\varphi^2 - \psi^2 \neq 0$ the last formula implies that $Z = 0$. The proof is completed.

Para-complex affine hyperspheres

Definition 5.

A nondegenerate para-complex hypersurface is said to be a *proper para-complex affine hypersphere* if there exists an affine normal vector field ζ such that $S = \alpha I$, where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\tau_2 = 0$.

If there exists an affine normal vector field ζ such that $S = 0$ and $\tau_2 = 0$ we say about an *improper para-complex affine hypersphere*.

Examples of para-complex affine hyperspheres

Example 1 Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x,y) := \frac{1}{2}\begin{pmatrix}\cos x\\ \sin x\\ \cos x\\ \sin x\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\cos y\\ \sin y\\ -\cos y\\ -\sin y\end{pmatrix}. \tag{27}$$

It easily follows that g is an immersion. Moreover $\widetilde{J}g_x = g_x$ and $\widetilde{J}g_y = -g_y$ so g is a para-holomorphic hypersurface. If we take $\zeta := -g$ then $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g. By straightforward computations we obtain

$$h_1 = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}, \quad h_2 = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{bmatrix}, \quad S = \mathrm{id}, \quad \tau_1 = \tau_2 = 0$$

relative to the canonical basis $\{\partial_x, \partial_y\}$.

Moreover, since

$$\theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \widetilde{J}\zeta] = \frac{1}{2}$$

one may easily compute that $H_\zeta = 1$ that is g is a proper para-complex affine sphere.

Example 2 Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x,y) := \frac{1}{2}\begin{pmatrix}\cosh x\\ \sinh x\\ \cosh x\\ \sinh x\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\cosh y\\ \sinh y\\ -\cosh y\\ -\sinh y\end{pmatrix}. \tag{28}$$

Exactly like in the previous example we have that g is an immersion and $\widetilde{J}g_x = g_x$ and $\widetilde{J}g_y = -g_y$ so g is a para-holomorphic hypersurface. Again taking $\zeta := -g$ we obtain that $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g. We also have

$$h_1 = \begin{bmatrix}-\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{bmatrix}, \quad h_2 = \begin{bmatrix}-\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}, \quad S = \mathrm{id}, \quad \tau_1 = \tau_2 = 0$$

relative to the canonical basis $\{\partial_x, \partial_y\}$.

Moreover, since

$$\theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \widetilde{J}\zeta] = \frac{1}{2}$$

we easily compute that $H_\zeta = 1$ that is g is a proper para-complex affine sphere.

Example 3 Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x,y) := \frac{1}{2}\begin{pmatrix} \cosh x \\ \sinh x \\ -\cosh x \\ -\sinh x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cos y \\ \sin y \\ \cos y \\ \sin y \end{pmatrix}. \tag{29}$$

It easily follows that g is an immersion and $\widetilde{J}g_x = g_x$ and $\widetilde{J}g_y = -g_y$ so g is a para-holomorphic hypersurface. If we take $\zeta := -g$ then $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g.

Example 4 Let $g\colon \mathbb{R}^2 \to \mathbb{R}^4$ be given by the formula

$$g(x,y) := \frac{1}{2}\begin{pmatrix} x \\ \frac{1}{2}x^2 \\ x \\ \frac{1}{2}x^2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} y \\ \frac{1}{2}y^2 \\ -y \\ -\frac{1}{2}y^2 \end{pmatrix}. \tag{30}$$

It easily follows that g is an immersion and $\widetilde{J}g_x = g_x$ and $\widetilde{J}g_y = -g_y$ so g is a para-holomorphic hypersurface. Let $\zeta := (0,0,0,1)^T$ then $\widetilde{J}\zeta = (0,1,0,0)^T$ and $\{\zeta, \widetilde{J}\zeta\}$ is a transversal bundle for g. We compute

$$h_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad h_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad S = 0, \quad \tau_1 = \tau_2 = 0$$

relative to the canonical basis $\{\partial_x, \partial_y\}$.

Since

$$\theta_\zeta(\partial_x, \partial_y) := \det[g_x, g_y, \zeta, \widetilde{J}\zeta] = -\frac{1}{2}$$

then $H_\zeta = -1$ that is g is an improper para-complex affine sphere.

Lemma 5.

Let $F\colon I \to \mathbb{R}^{2n}$ be a smooth function on interval I. If F satisfies the differential equation

$$F'(z) = -\widetilde{J}F(z), \tag{31}$$

then F is of the form

$$F(z) = \widetilde{J}v\cosh z - v\sinh z, \tag{32}$$

where $v \in \mathbb{R}^{2n}$.

Proof. It is not difficult to check, that functions of the form (32) satisfy differential equation (31). On the other hand, since (31) is a first order ordinary differential equation, the Picard-Lindelöf theorem implies that any solution of (31) must be of the form (32).

Theorem 3.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a centro-affine hypersurface with a $\widetilde{J}$-tangent centro-affine vector field. Then there exist an open subset $U \subset \mathbb{R}^{2n}$, an interval $I \subset \mathbb{R}$ and an immersion $g\colon U \to \mathbb{R}^{2n+2}$ such that f can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, z) = \widetilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z \qquad (33)$$

for all $(x_1, \ldots, x_{2n}, z) \in U \times I$.

Proof. Denote $C := -f$. Since f is centro-affine hypersurface with $\widetilde{J}$-tangent transversal vector field then we have $\widetilde{J}C = -\widetilde{J}f \in f_*(TM)$. Therefore for every $x \in M$ there exists a neighborhood V of x and a map $\psi(x_1, \ldots, x_{2n}, z)$ on V such that

$$f_* \frac{\partial}{\partial z} = \widetilde{J}C.$$

That is f can be locally expressed in the form $f(x_1, \ldots, x_{2n}, z)$, where $f_z = -\widetilde{J}f$. Now using the Lemma 5 we obtain the thesis.

Theorem 4.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with a centro-affine $\widetilde{J}$-tangent vector field $C = -\overrightarrow{of}$. If distribution $\mathcal{D}$ is involutive then for every $x \in M$ there exists a para-complex centro-affine immersion $g\colon V \to \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ such that f can be expressed in the neighborhood of x in the form

$$f(x_1,\ldots,x_{2n},z) = \widetilde{J}g(x_1,\ldots,x_{2n})\cosh z - g(x_1,\ldots,x_{2n})\sinh z. \quad (34)$$

Moreover, if $g\colon V \to \mathbb{R}^{2n+2}$ is a para-complex centro-affine immersion then f given by the formula (34) is an affine hypersurface with a centro-affine $\widetilde{J}$-tangent vector field and involutive distribution $\mathcal{D}$.

Proof. Let (φ, ξ, η) be an induced almost paracontact structure on M induced by C. The Frobenius theorem implies that for every $x \in M$ there exist an open neighborhood $U \subset M$ of x and linearly independent vector fields $X_1, \ldots, X_{2n}, X_{2n+1} = \xi \in \mathcal{X}(U)$ such that $[X_i, X_j] = 0$ for $i, j = 1, \ldots, 2n+1$. For every $i = 1, \ldots, 2n$ we have $X_i = D_i + \alpha_i \xi$ where $D_i \in \mathcal{D}$ and $\alpha_i \in C^\infty(U)$. Thus we have

$$0 = [X_i, \xi] = [D_i, \xi] - \xi(\alpha_i)\xi.$$

Now (9) and (12) imply that $[D_i, \xi]$ and $\xi(\alpha_i) = 0$. We also have

$$0 = [X_i, X_j] = [D_i, D_j] - D_j(\alpha_i)\xi + D_i(\alpha_j)\xi$$

for $i = 1, \ldots, 2n$.

Since $\mathcal{D}$ is involutive the above equality implies $[D_i, D_j] = 0$ for $i, j = 1, \ldots, 2n$. Of course the vector fields $D_1, \ldots, D_{2n}, \xi$ are linearly independent, so there exists a map $\psi(x_1, \ldots, x_{2n}, z)$ on U such that

$$\frac{\partial}{\partial z} = \xi, \quad \frac{\partial}{\partial x_i} = D_i, \quad i = 1, \ldots, 2n.$$

Now applying the Lemma 5 we find that f can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, z) = \widetilde{J} g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z$$

where $g \colon V \to \mathbb{R}^{2n+2}$ is an immersion defined on an open subset $V \subset \mathbb{R}^{2n}$. Moreover, since $\frac{\partial}{\partial x_i} \in \mathcal{D}$ we have that

$$f_{x_i} = \widetilde{J} g_{x_i} \cosh z - g_{x_i} \sinh z \in f_*(D).$$

Since $f_*(D)$ is $\widetilde{J}$ invariant we also have

$$\widetilde{J}f_{x_i} = g_{x_i}\cosh z - \widetilde{J}g_{x_i}\sinh z \in f_*(D).$$

The above implies that $g_{x_i} \in f_*(D)$ for $i = 1, \ldots, 2n$. But, since $\{g_{x_i}\}$ are linearly independent they form basis of $f_*(D)$ ($\dim f_*(D) = 2n$) so

$$f_*(D) = \operatorname{span}\{g_{x_1}, \ldots, g_{x_{2n}}\}.$$

Since $f_*(D)$ is $\widetilde{J}$-invariant we also have that

$$\widetilde{J}g_{x_i} \in f_*(D) = \operatorname{span}\{g_{x_1}, \ldots, g_{x_{2n}}\}$$

that is $\widetilde{J}g_{x_i} = \sum \alpha_i g_{x_i}$ where $\alpha_i \in C^\infty(U)$. Since g do not depend on variable z the functions α_i also do not, thus $\alpha_i \in C^\infty(V)$. In this way we have shown that for $g \colon V \to \mathbb{R}^{2n+2}$ the tangent space TV is $\widetilde{J}$-invariant (we can transfer $\widetilde{J}$ from $g_*(TV)$ to TV). Since $\widetilde{J}|_{f_*(D)}$ is para-complex and $f_*(D) = \text{span}_{C^\infty(U)}\{g_{x_1}, \ldots, g_{x_{2n}}\}$, so $\widetilde{J}$ is para-complex on TV. Finally g is para holomorphic. Since f is immersion $\{g_{x_1}, \ldots, g_{x_{2n}}, \widetilde{J}g\}$ are linearly independent. Moreover, since f is centro-affine we also have that g is linearly independent with $\{g_{x_1}, \ldots, g_{x_{2n}}, \widetilde{J}g\}$ that is $\{g, \widetilde{J}g\}$ forms $\widetilde{J}$-invariant transversal bundle to $g_*(TV)$. That is g is a para-complex affine immersion.

In order to prove the second part of the theorem note that since g is centro-affine para-complex affine immersion then $\{f_{x_1}, \ldots, f_{x_{2n}}, f_z, f\}$ are linearly independent. It means that f is an immersion and is centro-affine. Moreover, f is $\widetilde{J}$-tangent since $\widetilde{J}(-\overrightarrow{of}) = -g\cosh z + \widetilde{J}g\sinh z = f_z$. In particular g is para holomorphic that is we have $\widetilde{J}g_{x_i} = \sum_{j=1}^{2n} \alpha_{ij} g_{x_j}$ for $i = 1, \ldots, 2n$. Now by straightforward computations we get $\sum_{j=1}^{2n} \alpha_{ij} f_{x_j} = \widetilde{J}f_{x_i}$ for $i = 1, \ldots, 2n$. That is $\widetilde{J}f_{x_i} \in \operatorname{span}\{f_{x_1}, \ldots, f_{x_{2n}}\}$. In this way we have shown that $\operatorname{span}\{f_{x_1}, \ldots, f_{x_{2n}}\}$ is $\widetilde{J}$-invariant and since its dimension is $2n$ it must be equal to $f_*(D)$. Now it is easy to see that $\mathcal{D} = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}}\}$ is involutive as generated by canonical vector fields.

$\widetilde{J}$-tangent affine hyperspheres

Theorem 5.

There are no improper $\widetilde{J}$-tangent affine hyperspheres.

Proof. We have $\eta(SX) = -h(X, \xi)$ for all $X \in \mathcal{X}(M)$. Thus, if $S = 0$, $h(X, \xi) = 0$ for every $X \in \mathcal{X}(M)$, which contradicts nondegeneracy of h.

Classification of $\widetilde{J}$-tangent affine hyperspheres

Theorem 6.

Let $f\colon M \to \mathbb{R}^{2n+2}$ be a $\widetilde{J}$-tangent affine hypersphere with an involutive distribution $\mathcal{D}$. Then f can be locally expressed in the form:

$$f(x_1, \ldots, x_{2n}, z) = \widetilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z \tag{35}$$

where g is a proper para-complex affine hypersphere. Moreover, the converse is also true in the sense that if g is a proper para-complex affine hypersphere then f given by the formula (35) is a $\widetilde{J}$-tangent affine hypersphere with an involutive distribution $\mathcal{D}$.

Proof. ($\Rightarrow$) First note that due to Theorem 5 f must be a proper affine hypersphere. Let C be a $\widetilde{J}$-tangent affine normal field. There exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $C = -\lambda f$. Since C is $\widetilde{J}$-tangent and transversal the same is $\frac{1}{\lambda}C = -f$. Thus f satisfies assumptions of Theorem 4. By Theorem 4 there exists a para-complex centro-affine immersion $g\colon V \to \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ and there exists an open interval I such that f can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, z) = \widetilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z \quad (36)$$

for $(x_1, \ldots, x_{2n}) \in V$ and $z \in I$.

Let $\zeta := -|\lambda|^{\frac{2n+3}{2n+4}} g$ and let $\nabla, h_1, h_2, S, \tau_1, \tau_2$ be induced objects on V by ζ. Using the Weingarten formula for g and ζ we get

$$\mathrm{D}_{\partial_{x_i}} \zeta = -g_*(S\partial_{x_i}) + \tau_1(\partial_{x_i})\zeta + \tau_2(\partial_{x_i})J\zeta.$$

On the other hand, by straightforward computations we have

$$\mathrm{D}_{\partial_{x_i}} \zeta = \partial_{x_i}(\zeta) = -|\lambda|^{\frac{2n+3}{2n+4}} g_*(\partial_{x_i}).$$

Thus we obtain

$$S = |\lambda|^{\frac{2n+3}{2n+4}} I, \quad \tau_1 = 0, \quad \tau_2 = 0. \tag{37}$$

Now, to prove that ζ is an affine normal vector field it is enough to show that $|H_\zeta| = 1$. Since g is para-holomorphic, without loss of generality, we may assume that

$$\partial_{x_{n+i}} = \widetilde{J}\partial_{x_i}$$

for $i = 1 \ldots n$. Let

$$a := \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_n}, \widetilde{J}\partial_{x_1}, \ldots, \widetilde{J}\partial_{x_n}).$$

Then

$$\frac{1}{a}\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \widetilde{J}\partial_{x_1}, \ldots, \widetilde{J}\partial_{x_n}$$

is a unimodular basis relative to θ_ζ.

Now

$$H_\zeta = \frac{1}{a^2}\det\begin{bmatrix} h_1(\partial_{x_1},\partial_{x_1}) & h_1(\partial_{x_1},\partial_{x_2}) & \cdots & h_1(\partial_{x_1},\partial_{x_{2n}}) \\ h_1(\partial_{x_2},\partial_{x_1}) & h_1(\partial_{x_2},\partial_{x_2}) & \cdots & h_1(\partial_{x_2},\partial_{x_{2n}}) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(\partial_{x_{2n}},\partial_{x_1}) & h_1(\partial_{x_{2n}},\partial_{x_2}) & \cdots & h_1(\partial_{x_{2n}},\partial_{x_{2n}}) \end{bmatrix}.$$

By the Gauss formula for g we have

$$\begin{aligned} g_{x_ix_j} &= g_*(\nabla_{\partial_{x_i}}\partial_{x_j}) + h_1(\partial_{x_i},\partial_{x_j})\zeta \\ &\quad + h_2(\partial_{x_i},\partial_{x_j})\widetilde{J}\zeta \qquad (38) \\ &= g_*(\nabla_{\partial_{x_i}}\partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i},\partial_{x_j})g - |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial_{x_i},\partial_{x_j})\widetilde{J}g. \qquad (39) \end{aligned}$$

Let $\overline{\nabla}$ and $\overline{h}$ be an induced connection and the second fundamental form for f. By the Gauss formula for f we have

$$
\begin{aligned}
f_{x_i x_j} &= \widetilde{J} g_{x_i x_j} \cosh z - g_{x_i x_j} \sinh z \\
&= f_*(\overline{\nabla}_{\partial_{x_i}} \partial_{x_j}) && (40) \\
&- \lambda \overline{h}(\partial_{x_i}, \partial_{x_j})(\widetilde{J} g \cosh z - g \sinh z). && (41)
\end{aligned}
$$

Applying (38) to (40) we get

$$
\begin{aligned}
f_*(\overline{\nabla}_{\partial_{x_i}} \partial_{x_j}) - \lambda \overline{h}(\partial_{x_i}, \partial_{x_j})(\widetilde{J} g \cosh z - g \sinh z) \\
= \widetilde{J} g_*(\nabla_{\partial_{x_i}} \partial_{x_j}) \cosh z - g_*(\nabla_{\partial_{x_i}} \partial_{x_j}) \sinh z \\
- |\lambda|^{\frac{2n+3}{2n+4}} (h_1(\partial_{x_i}, \partial_{x_j}) \widetilde{J} g + h_2(\partial_{x_i}, \partial_{x_j}) g) \cosh z \\
+ |\lambda|^{\frac{2n+3}{2n+4}} (h_1(\partial_{x_i}, \partial_{x_j}) g + h_2(\partial_{x_i}, \partial_{x_j}) \widetilde{J} g) \sinh z \\
= f_*(\nabla_{\partial_{x_i}} \partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j})(\widetilde{J} g \cosh z - g \sinh z) \\
- |\lambda|^{\frac{2n+3}{2n+4}} h_2(\partial_{x_i}, \partial_{x_j})(g \cosh z - \widetilde{J} g \sinh z) \\
= f_*(\nabla_{\partial_{x_i}} \partial_{x_j}) - |\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j}) \cdot f
\end{aligned}
$$

Since $f_*(\nabla_{\partial_{x_i}}\partial_{x_j})$ as well as $\widetilde{J}f$ are tangent we immediately obtain

$$-\lambda h(\partial_{x_i}, \partial_{x_j}) = -|\lambda|^{\frac{2n+3}{2n+4}} h_1(\partial_{x_i}, \partial_{x_j}).$$

By the Gauss formula for f we also have

$$h(\partial_z, \partial_z) = -\frac{1}{\lambda}$$

and

$$h(\partial_z, \partial_{x_i}) = h(\partial_{x_i}, \partial_z) = 0$$

for $i = 1 \ldots 2n$.

Hence

$$
\det h := \begin{bmatrix} h(\partial_{x_1},\partial_{x_1}) & h(\partial_{x_1},\partial_{x_2}) & \cdots & h(\partial_{x_1},\partial_{x_{2n}}) & 0 \\ h(\partial_{x_2},\partial_{x_1}) & h(\partial_{x_2},\partial_{x_2}) & \cdots & h(\partial_{x_2},\partial_{x_{2n}}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(\partial_{x_{2n}},\partial_{x_1}) & h(\partial_{x_{2n}},\partial_{x_2}) & \cdots & h(\partial_{x_{2n}},\partial_{x_{2n}}) & 0 \\ 0 & 0 & \cdots & 0 & -\frac{1}{\lambda} \end{bmatrix}
$$

$$
= -\frac{1}{\lambda}\det[h(\partial_{x_i},\partial_{x_j})] = -\frac{1}{\lambda}\cdot\left(\frac{1}{\lambda}\cdot|\lambda|^{\frac{2n+3}{2n+4}}\right)^{2n}\det[h_1(\partial_{x_i},\partial_{x_j})]
$$

$$
= -\frac{1}{\lambda}\cdot|\lambda|^{-\frac{2n}{2n+4}}a^2 H_\zeta .
$$

Now

$$(\omega_h)^2 = |\det h| = |\lambda|^{\frac{-2n-2}{n+2}} a^2 |H_\zeta| \tag{42}$$

On the other hand we have

$$\begin{aligned}\omega_h &= \theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z) = \det[f_{x_1}, \ldots, f_{x_{2n}}, f_z, C] \\ &= -\lambda \det[\widetilde{J} g_{x_1} \cosh z - g_{x_1} \sinh z, \ldots, \widetilde{J} g_{x_{2n}} \cosh z - g_{x_{2n}} \sinh z, \\ &\qquad \widetilde{J} g \sinh z - g \cosh z, \widetilde{J} g \cosh z - g \sinh z].\end{aligned}$$

Since determinant is $(2n+2)$-linear and antisymmetric and since $g_{x_{n+i}} = \widetilde{J} g_{x_i}$ for $i = 1 \ldots n$ eventually we obtain

$$\begin{aligned}\omega_h &= -\lambda \det[g_{x_1}, \ldots, g_{x_n}, \widetilde{J} g_{x_1}, \ldots, \widetilde{J} g_{x_n}, g, \widetilde{J} g] \\ &= -\lambda (|\lambda|^{\frac{2n+3}{2n+4}})^{-2} \det[g_*(\partial_{x_1}), \ldots, g_*(\partial_{x_{2n}}), \zeta, \widetilde{J}\zeta] \\ &= -\lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}}) \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}}) = -\lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}}) a.\end{aligned}$$

Using the above formula in (42) we easily obtain

$$|H_\zeta| = a^{-2}|\lambda|^{\frac{2n+2}{n+2}} \cdot \lambda^2 \cdot |\lambda|^{-\frac{4n+6}{n+2}} a^2 = 1.$$

("$\Leftarrow$") Let $g\colon U \to \mathbb{R}^{2n+2}$ be a proper para-complex affine hypersphere. Since g is a proper para-complex affine hypersphere there exists $\alpha \neq 0$ such that $\zeta = -\alpha g$ is an affine normal vector field. Without loss of generality we may assume that $\alpha > 0$. Of course both g and $\widetilde{J}g$ are transversal thus $\{g_{x_1}, \ldots, g_{x_{2n}}, g, \widetilde{J}g\}$ form the basis of $\mathbb{R}^{2n+2}$. The above implies that

$$f\colon U \times I \ni (x_1, \ldots, x_{2n}, z) \mapsto f(x_1, \ldots, x_{2n}, z) \in \mathbb{R}^{2n+2}$$

given by the formula:

$$f(x_1, \ldots, x_{2n}, z) := \widetilde{J}g(x_1, \ldots, x_{2n}) \cosh z - g(x_1, \ldots, x_{2n}) \sinh z$$

is an immersion and $C := -\alpha^{\frac{2n+4}{2n+3}} \cdot f$ is a transversal vector field.

Field C is $\widetilde{J}$-tangent because $\widetilde{J}C = \alpha^{\frac{2n+4}{2n+3}} f_z$. Since C is equiaffine it is enough to show that $\omega_h = \theta$ for some positively oriented (relative to θ) basis on $U \times I$. Let $\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z$ be a local coordinate system on $U \times I$. Since g is para-holomorphic we may assume that $\partial_{x_{n+i}} = \widetilde{J}\partial_{x_i}$ for $i = 1 \ldots n$.
Then we have

$$
\begin{aligned}
\theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z) &= \det[f_{x_1}, \ldots, f_{x_{2n}}, f_z, -\alpha^{\frac{2n+4}{2n+3}} f] \\
&= -\alpha^{\frac{2n+4}{2n+3}} \det[\widetilde{J}g_{x_1} \cosh z - g_{x_1} \sinh z, \ldots, \widetilde{J}g_{x_{2n}} \cosh z - g_{x_{2n}} \sinh z, \\
&\qquad + \widetilde{J}g \sinh z - g \cosh z, \widetilde{J}g \cosh z - g \sinh z] \\
&= -\alpha^{\frac{2n+4}{2n+3}} \det[g_*(\partial_{x_1}), \ldots, g_*(\partial_{x_{2n}}), g, \widetilde{J}g] \\
&= -\alpha^{\frac{2n+4}{2n+3}} \cdot \frac{1}{\alpha^2} \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}}) \\
&= -\alpha^{-\frac{2n+2}{2n+3}} \theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}}).
\end{aligned}
$$

In a similar way, like in the proof of the first implication we compute

$$\begin{aligned}\det h &= \alpha^{-\frac{2n+4}{2n+3}} \cdot \left(\frac{\alpha}{\alpha^{\frac{2n+4}{2n+3}}}\right)^{2n} \det h_1 \\ &= \alpha^{-\frac{2n+4}{2n+3}} \cdot \alpha^{-\frac{2n}{2n+3}} \det h_1 \\ &= \alpha^{\frac{-4n-4}{2n+3}} \det h_1 .\end{aligned}$$

The above implies that

$$(\omega_h)^2 = |\det h| = \alpha^{\frac{-4n-4}{2n+3}} |\det h_1|.$$

Since

$$|\det h_1| = |H_\zeta| [\theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})]^2$$

we obtain

$$(\omega_h)^2 = \alpha^{\frac{-4n-4}{2n+3}} |H_\zeta| [\theta_\zeta(\partial_{x_1}, \ldots, \partial_{x_{2n}})]^2.$$

Finally, using the fact that $|H_\zeta| = 1$, we get $\omega_h = |\theta(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z)|$. The proof is completed.

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Thank you!