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Semi-discrete constant mean curvature surfaces of revolution in Minkowski space

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Partly joint work with Christian Müller (TU Wien)

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Geoemtry, Integrability and Quantization

Contents

References

[1] F. Burstall, U. Hertrich-Jeromin, C. Müller and W. Rossman, *Semi-discrete isothermic surfaces*, to appear in Geometriae Dedicata.

[2] C. Müller and J. Wallner, *Semi-discrete isothermic surfaces*, Results in Math. **63** (2013), no. 3-4, 1395-1407.

[3] W. Rossman and M. Yasumoto, *Weierstrass representation for semi-discrete minimal surfaces, and comparison of various discretized catenoids*, Journal of Math-for-Industry **4B** (2012) 109-118.

[4] M. Yasumoto, *Semi-discrete maximal surfaces with singularities in Minkowski space*, preprint.

Symbols

(A, B, C, D) : quadrilateral with vertices A, B, C, D
$\mathbb{R}^3$: Euclidean 3-space
$\mathbb{R}^{n,1}$: Minkowski $(n+1)$-space with the Lorentz metric

$$\begin{aligned}\langle (x_1, x_2, \cdots, x_n, x_0)^t, (y_1, y_2, \cdots, y_n, y_0)^t \rangle \\ = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n - x_0 y_0,\end{aligned}$$

for $(x_1, x_2, \cdots, x_n, x_0)^t,\ (y_1, y_2, \cdots, y_n, y_0)^t \in \mathbb{R}^{n,1}$
$\mathbb{C}$: complex plane
$\mathbb{S}^1$: unit circle in $\mathbb{C}$

Abbreviation

$$x = x(k, t)\ ((k, t) \in \mathbb{Z} \times \mathbb{R}),\ x_1 = x(k+1, t),$$
$$x' = \frac{\partial x}{\partial t},\ \Delta x = x_1 - x$$

1 Introduction

What is *discrete differential geometry* (DDG) (for me)?

Discretization of smooth objects (curves, surfaces)

$\rightarrow$ reconstruct differential geometry in the discrete setting by using integrable systems techniques, differential geometric techniques, etc

Examples.

discretization of curves

Hoffmann, 2004, etc...

Inoguchi, Kajiwara, Matsuura, Ohta, 2012, 2014, etc...

discretization of surfaces

Bobenko, Pinkall, 1996, 1998, etc...

Burstall, Hertrich-Jeromin, Rossman, 2008, 2012, etc...

Examples of discrete surfaces

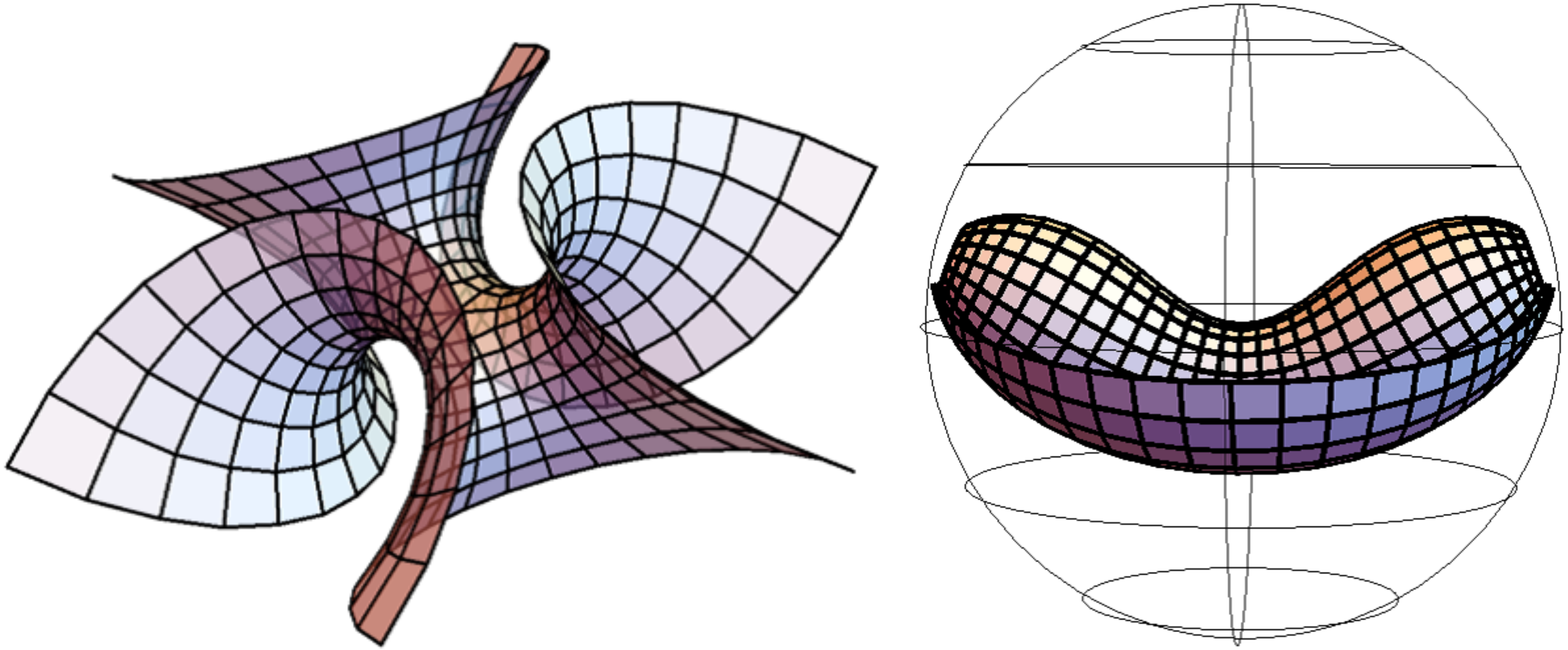

Fig.1 Left-hand picture: a discrete higher order Enneper surface in $\mathbb{R}^3$. Right-hand picture: a discrete CMC 1 Enneper cousin in $\mathbb{H}^3$.

semi-discretization of surfaces

Müller, Wallner, 2011, Rossman, Y-, 2012,
Y-, 2015, Burstall, Jeromin, Müller, Rossman, 2015...

Examples of semi-discrete surfaces

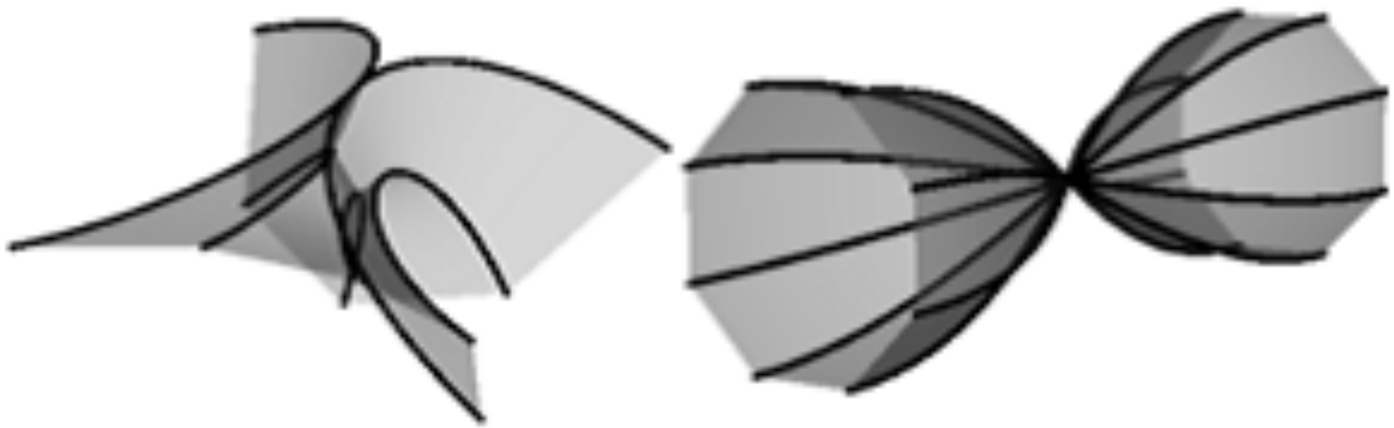

Fig.2 Left-hand picture: a semi-discrete Enneper surface in $\mathbb{R}^3$. Right-hand picture: a semi-discrete constant positive Gaussian curvature surface in $\mathbb{R}^3$.

Why semi-discretize?

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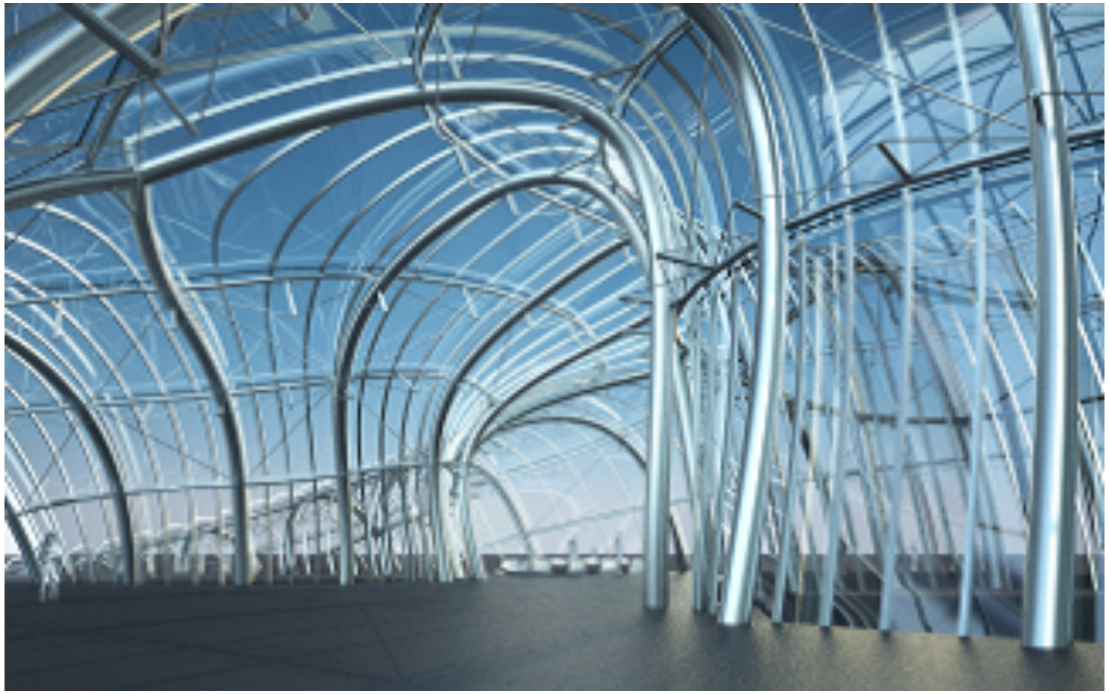
1. Applicable to the real world



(Pottmann, Schiftner, Bo, etc...)

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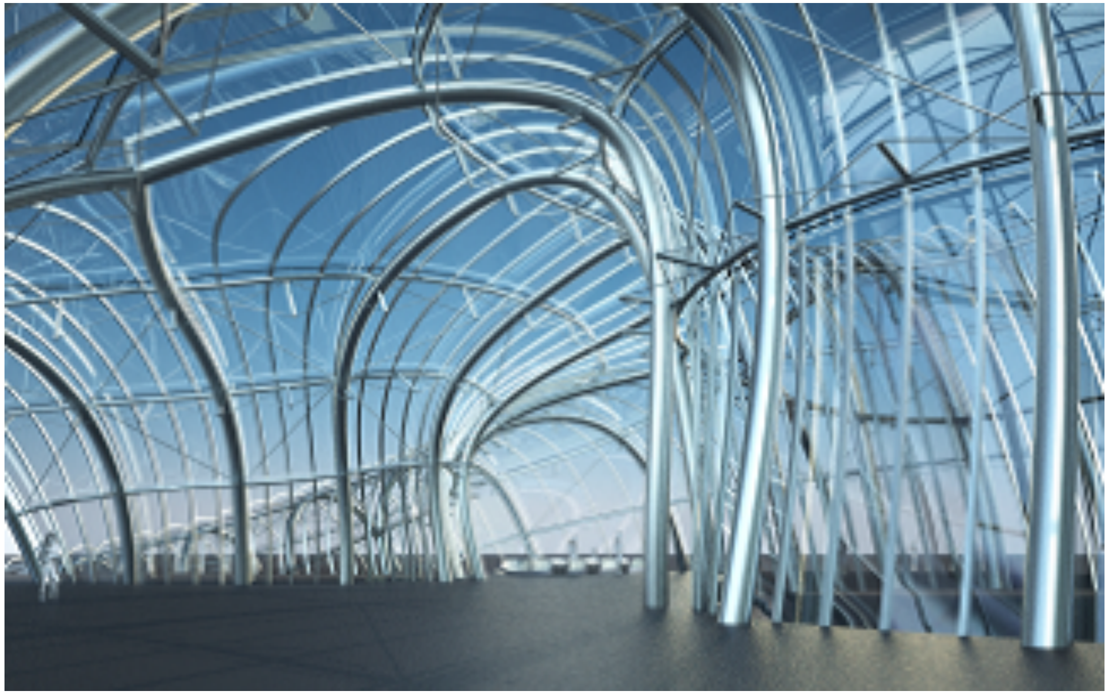


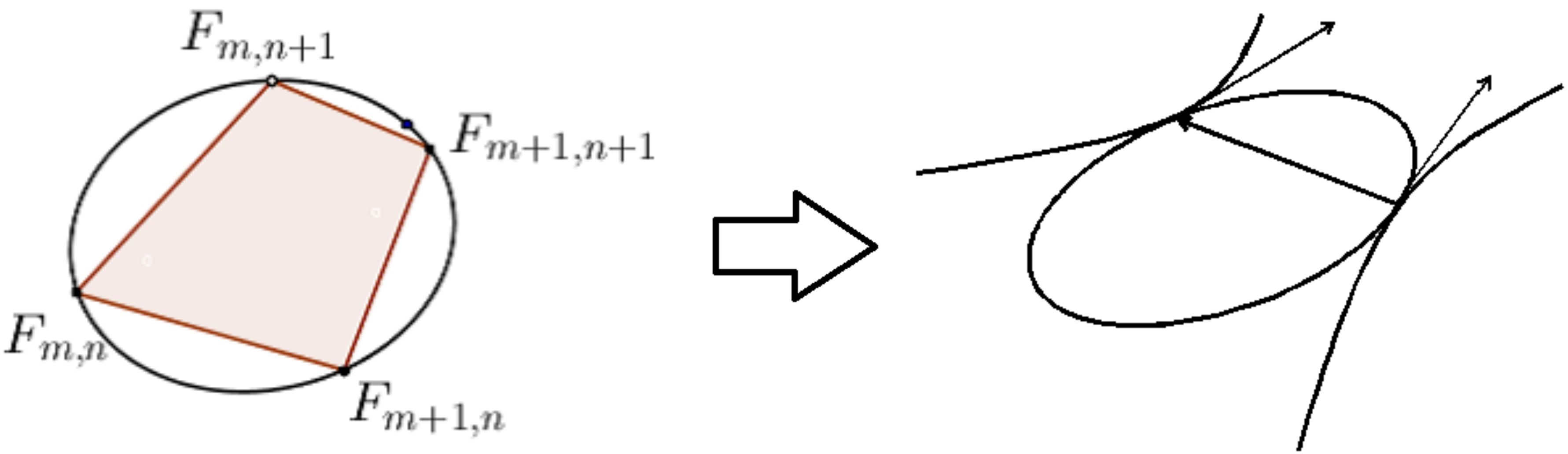
(Pottmann, Schiftner, Bo, etc...)

2. Bridge between discrete and smooth surfaces

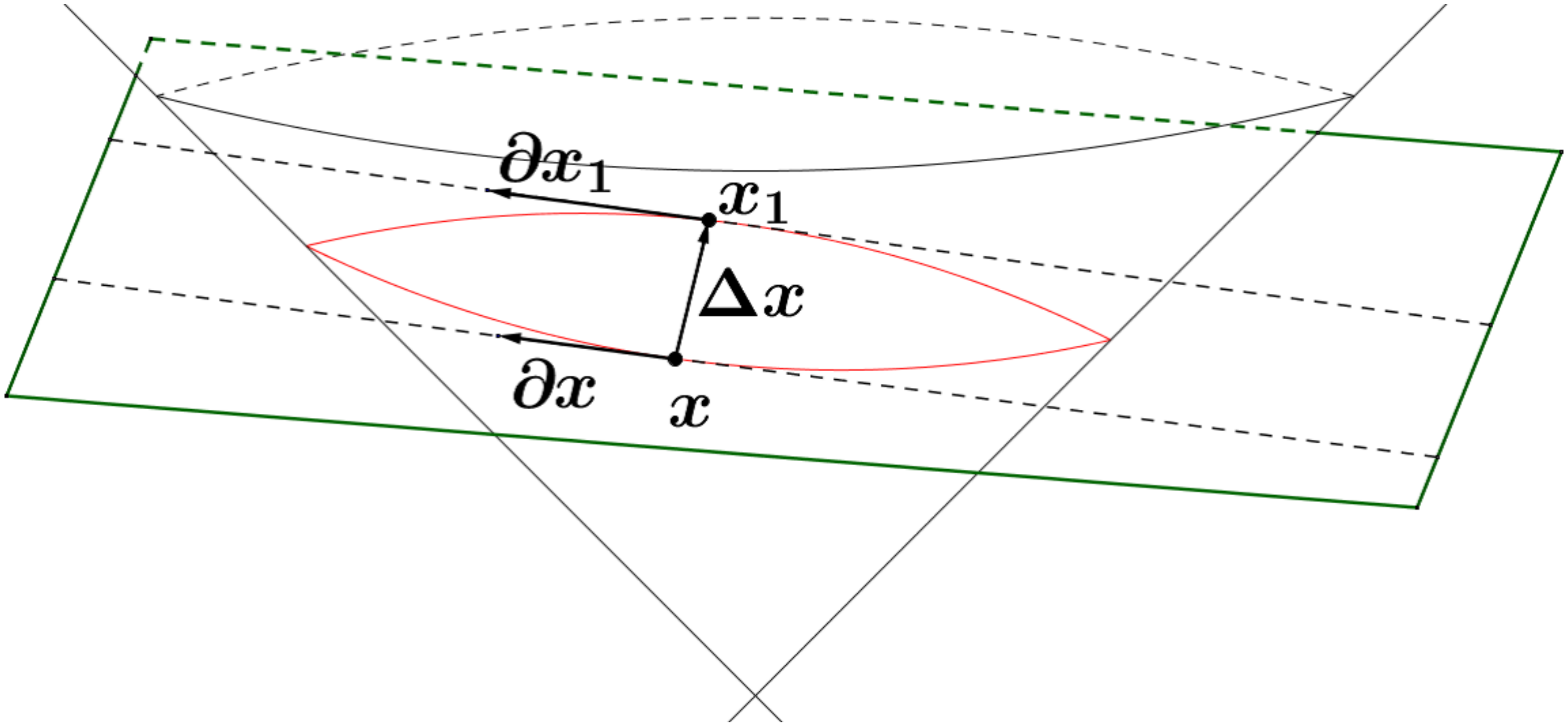
$\rightarrow$It might be helpful to understand the similarities or differences between discrete and smooth surfaces.

2 Semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$

semi-discrete surfaces :=: surfaces discretized in only one of the two parameter directions
(or surfaces obtained by taking limit in only one of the two discrete parameters of a fully-discrete surfaces)



Let $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^{2,1}$ be a semi-discrete surface. Then "circularity" of semi-discrete surfaces is as follows.



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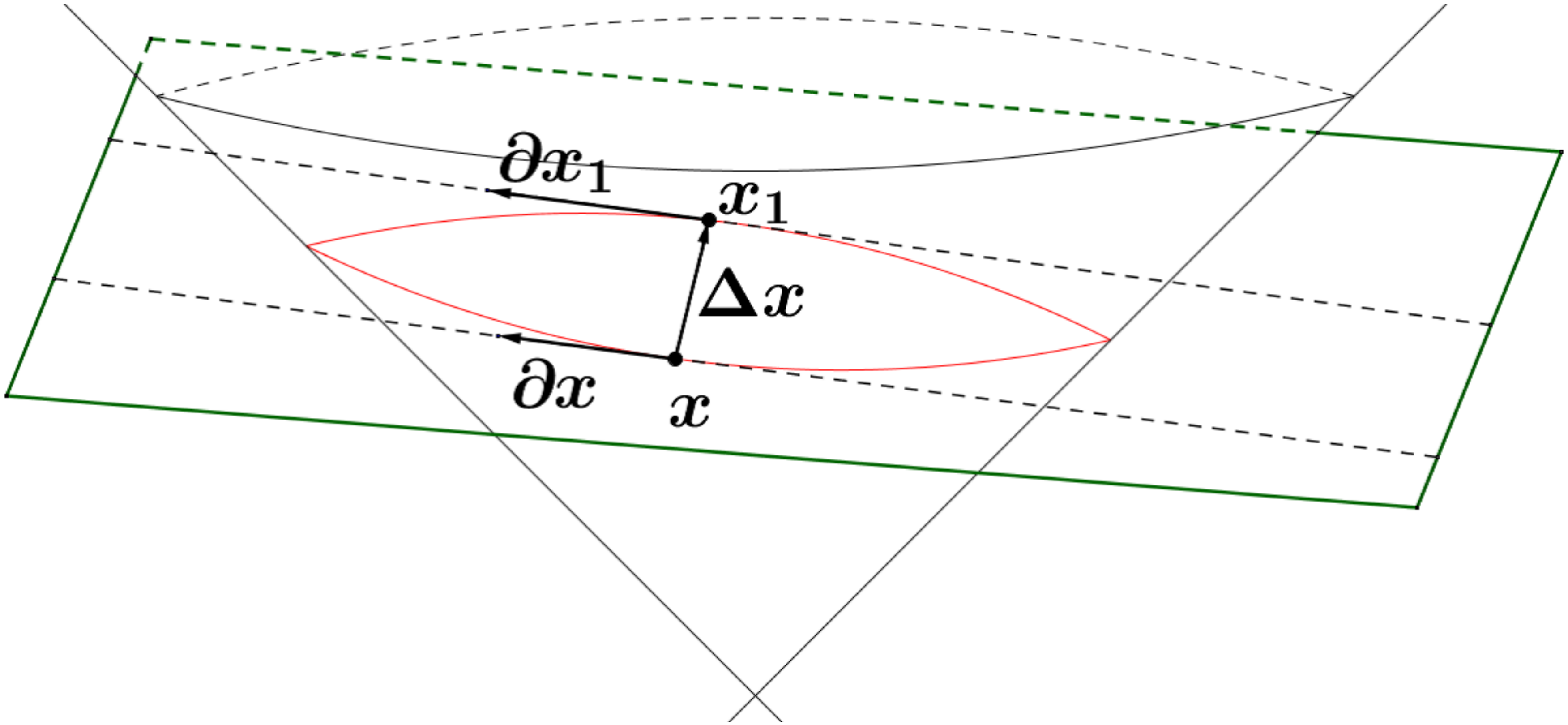

I.e. there exists an intersection of a plane and translated light cone $\mathscr{C}$ passing through x and x_1 that is tangent to x', x_1' there (for all $(k, t) \in \mathbb{Z} \times \mathbb{R}$).

Def 1. • A circular semi-discrete surface x is *isothermic* if there exist positive functions $\nu,\ \sigma,\ \tau$ such that

$$\|\Delta x\|^2 = \sigma\nu\nu_1,\ \|x'\|^2 = \tau\nu^2,\ \text{with } \sigma' = \Delta\tau = 0.$$

• Let x be a semi-discrete isothermic surface. Then x^* is a *dual surface* of x if there exists a function $\nu : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^+$ so that

$$(x^*)' = -\frac{1}{\nu^2}x',\ \Delta x^* = \frac{1}{\nu\nu_1}\Delta x.$$

• A semi-discrete isothermic surface x is a *semi-discrete maximal surface* if the "mean curvature" H of x identically vanishes.

Thm 1. Any semi-discrete maximal surface x can be **locally** constructed using a semi-discrete holomorphic function g by solving

$$x' = -\frac{\tau}{2}\mathrm{Re}\left(\frac{1+g^2}{g'}, \frac{i(1-g^2)}{g'}, -\frac{2g}{g'}\right)^t,$$

$$\Delta x = \frac{\sigma}{2}\mathrm{Re}\left(\frac{1+gg_1}{\Delta g}, \frac{i(1-gg_1)}{\Delta g}, -\frac{g+g_1}{\Delta g}\right)^t$$

with τ and σ determined from g.

Remark. $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^2 \cong \mathbb{C}$ is a *semi-discrete holomorphic function* if it is semi-discrete isothermic. $\square$

Examples.

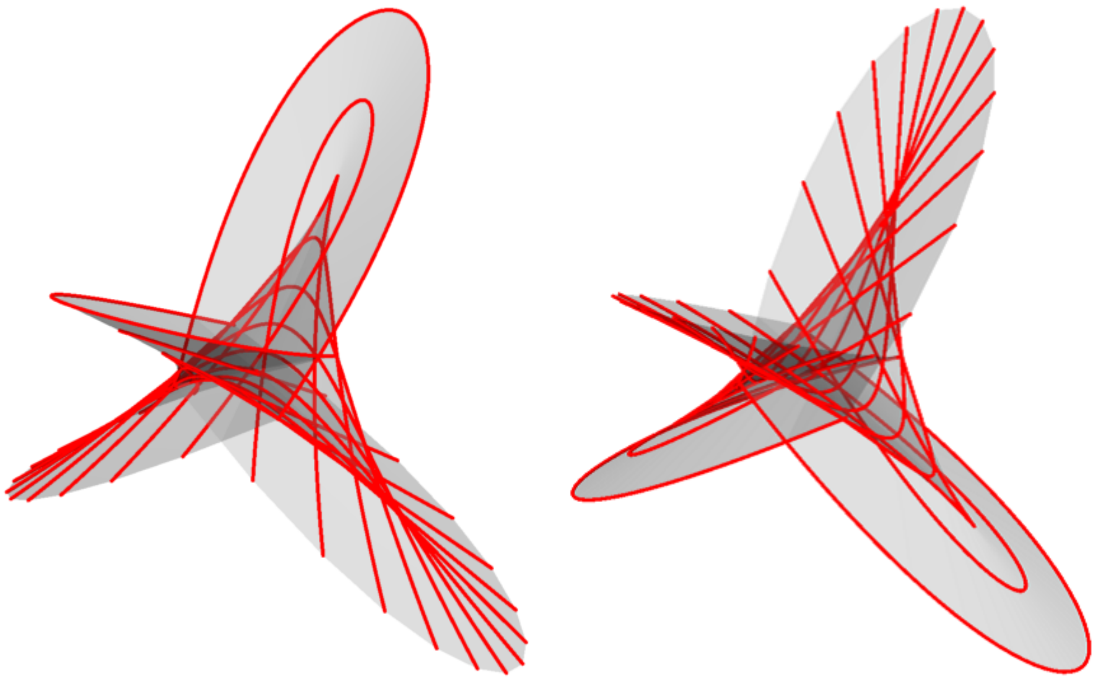

Fig.3 Two types of semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$ with semi-discrete holomorphic function $g(k,t) = c_1(t+ik)$ and $g(k,t) = c_1(k+it)$ $(c_1 \in \mathbb{R})$

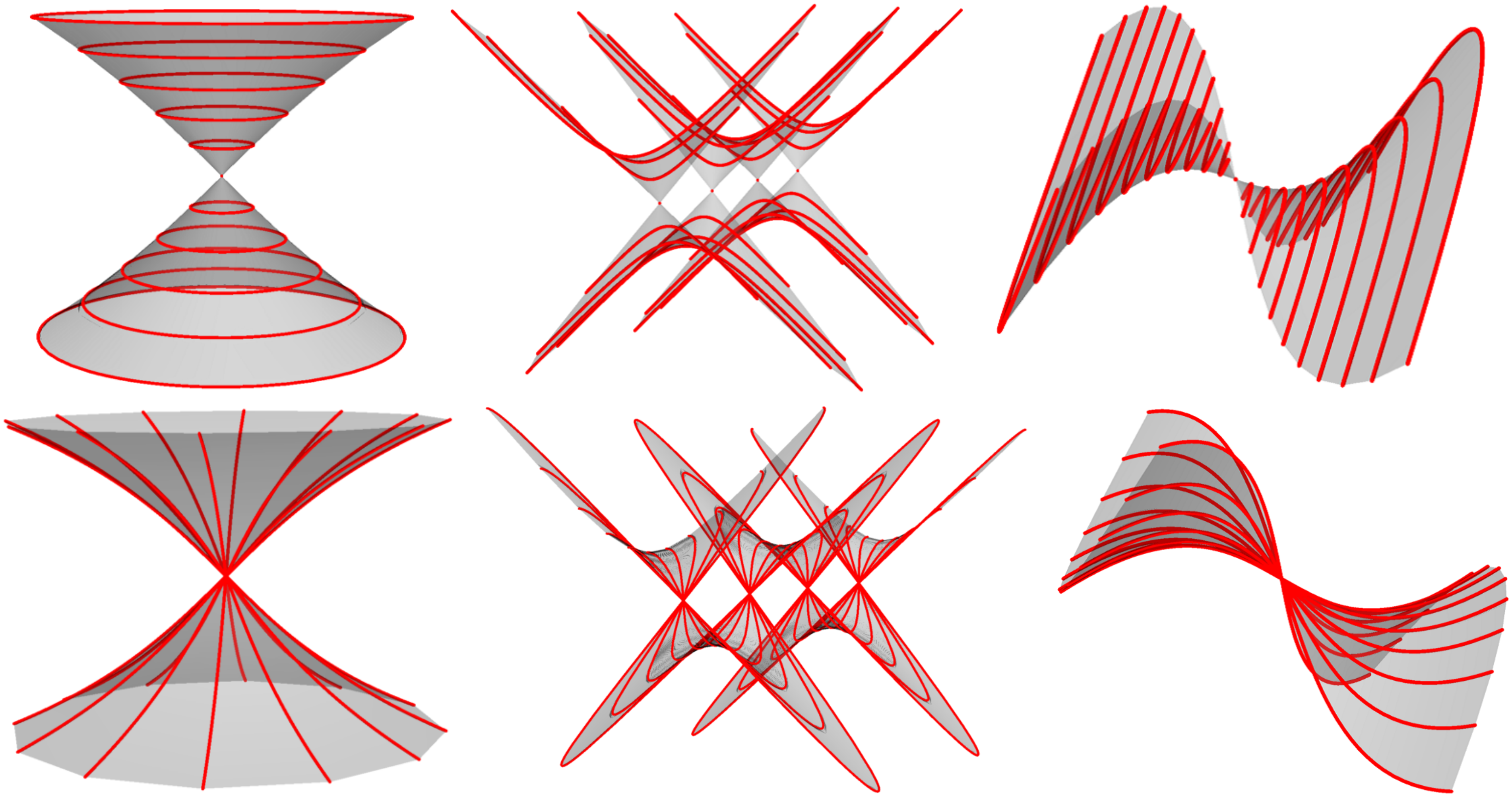

Fig.4 Semi-discrete maximal surfaces of revolution in $\mathbb{R}^{2,1}$ with timelike, spacelike and lightlike axes

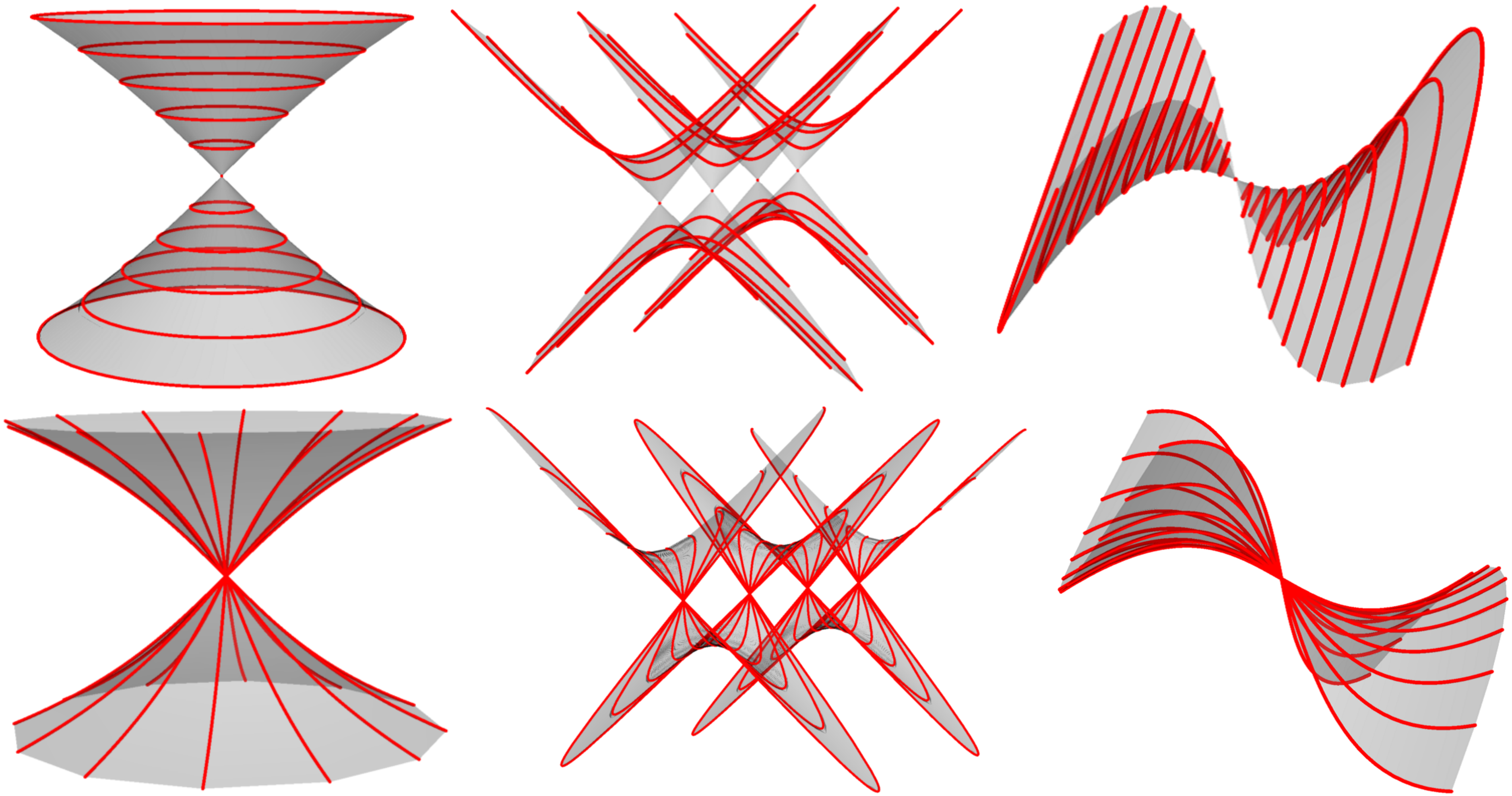

Fig.4 Semi-discrete maximal surfaces of revolution in $\mathbb{R}^{2,1}$ with timelike, spacelike and lightlike axes
$\rightarrow$They seem to have “singularities”.

Singularities of semi-discrete maximal surfaces are defined as follows:

Def 2. Let x be a semi-discrete maximal surface. Then a edge $[x, x_1]$ is a *singular edge* if the plane spanned by $x',\ x_1',\ \Delta x$ is not spacelike.

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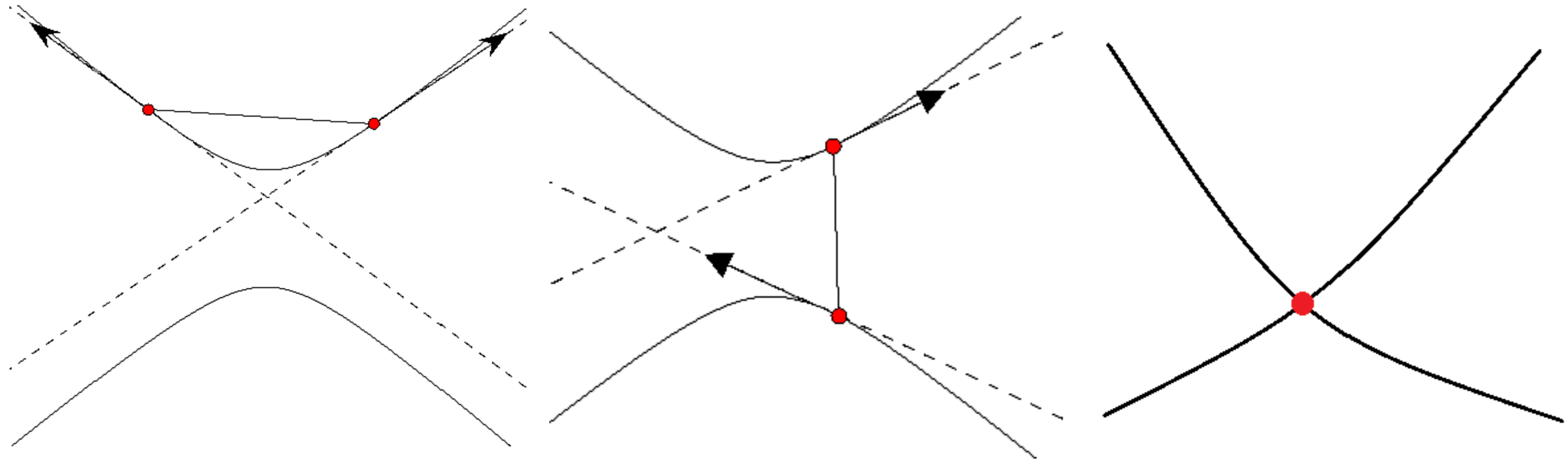
Def 2. Let x be a semi-discrete maximal surface. Then a edge $[x, x_1]$ is a *singular edge* if the plane spanned by $x',\ x'_1,\ \Delta x$ is not spacelike.

Motivation

- We want the corresponding plane to not lie in a spacelike plane, because of the behavior of the smooth case.
- We hope that singularities of semi-discrete maximal surfaces will appear exactly when the image of the corresponding semi-discrete holomorphic Gauss map g lies near $\mathbb{S}^1$.

Thm 2. Let $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ be a semi-discrete holomorphic function and let x be a semi-discrete maximal surface determined from g. Then the edge $[x, x_1]$ is a singular edge if and only if $\mathscr{C}$ intersects $\mathbb{S}^1$.



From left to right: an embedded generic singular edge, a twisted generic singular edge, a non-generic singular edge

3 Semi-discrete non-zero CMC surfaces of revolution in $\mathbb{R}^{2,1}$ (with Müller)

Our goal

Define and analyze singularities of semi-discrete CMC surfaces in $\mathbb{R}^{2,1}$

Our hope

The definition of "singularities" is the same as the singular edges of semi-discrete maximal surfaces.

But we do not have any example of semi-discrete CMC surfaces in $\mathbb{R}^{2,1}$. So our first task is to make the examples.

The following fact is known.

Prop 1. Let x be a semi-discrete isothermic surface in $\mathbb{R}^{2,1}$ and let H be the "mean curvature" of x. Then

$$H \equiv \text{constant} \neq 0 \Leftrightarrow x^* = x + \frac{1}{H} n, \text{ where}$$

n is a map from $\mathbb{Z} \times \mathbb{R}$ to $\mathbb{H}^2 := \{X \in \mathbb{R}^{2,1} | \langle X, X \rangle = -1\}$ satisfying $n' \parallel x', \ \Delta n \parallel \Delta x$.

Using the above proposition, we can compute the explicit parametrizations of semi-discrete CMC surfaces of revolution with **smooth profile curve** in $\mathbb{R}^{2,1}$.

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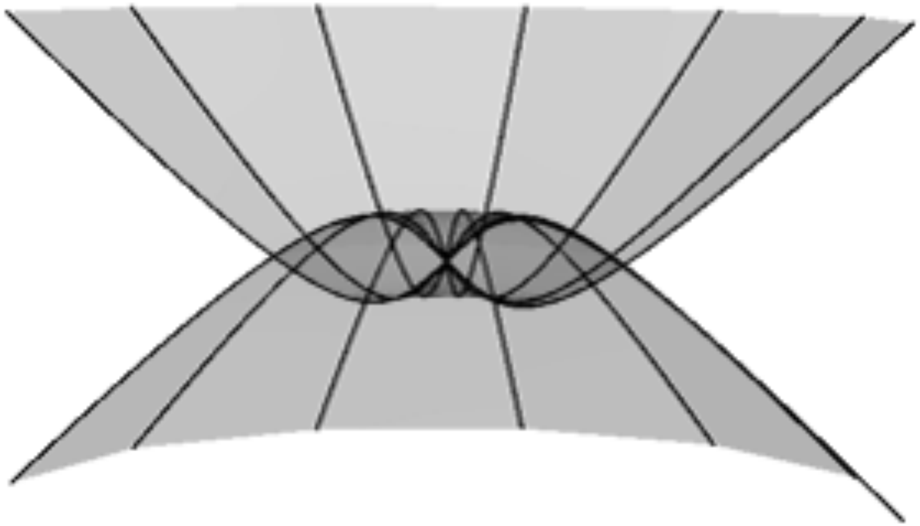
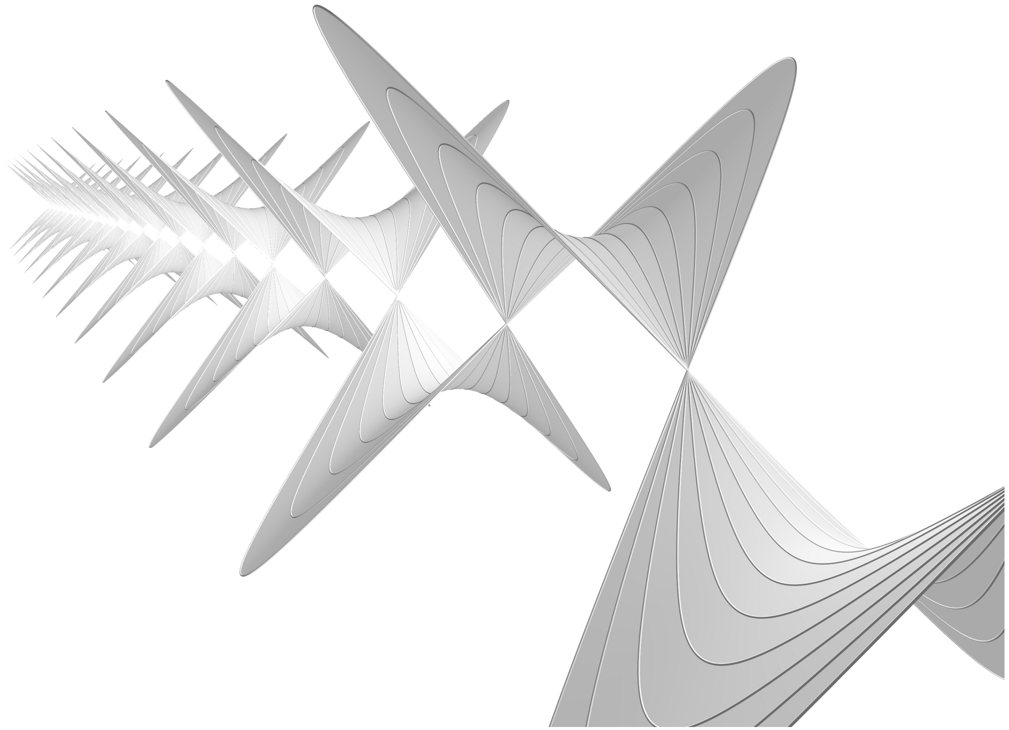
Using the above proposition, we can compute the explicit parametrizations of semi-discrete CMC surfaces of revolution with **smooth profile curve** in $\mathbb{R}^{2,1}$.
Here we state only the result for profile curves of semi-discrete CMC surfaces of revolution in $\mathbb{R}^{2,1}$.

Thm 3. Semi-discrete CMC surfaces of revolution with smooth profile curves have the same collection of profile curves as the smooth CMC surfaces of revolution.





We can analyze singularities of semi-discrete CMC surfaces of revolution with smooth profile curve.

Singular edges of semi-discrete CMC surfaces of revolution appear around the cone point.

However, embedded singular edges might also appear when the profile curve comes close to a lightlike line asymptotically.
$\rightarrow$A singular edges is one of the possibilities of "singularities" of semi-discrete CMC surfaces in $\mathbb{R}^{2,1}$.

On the other hand, there is no non-generic singular edge like cone point.

Thm 4. Away from generic singular edges, semi-discrete CMC (or, maximal) surfaces of revolution with smooth profile curve have only one non-generic singular edge.

Summary

- We have a Weierstrass-type representation for semi-discrete maximal surfaces in $\mathbb{R}^{2,1}$.
- We have explicit parametrizations of semi-discrete CMC surfaces of revolution with smooth profile curve in $\mathbb{R}^{2,1}$.

Problems

- Constructing semi-discrete CMC surfaces of revolution with smooth profile curve in $\mathbb{R}^{2,1}$
- Construction method of semi-discrete CMC surfaces in $\mathbb{R}^{2,1}$
- Singularities of semi-discrete CMC surfaces

Thank you very much.