Holomorphically Projective Mappings

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Topics

1. Introduction
2. Kähler manifolds
3. Holomorphically projective mapping theory for $K_n \to \bar{K}_n$ of class C^1
4. Holomorphically projective mapping theory for $K_n \to \bar{K}_n$ of class C^2
5. Holomorphically projective mappings between $K_n \in C^r (r > 2)$ and $\bar{K}_n \in C^2$

1. Introduction

We study the general dependence of holomorphically projective mappings of classical, pseudo- and hyperbolic Kähler manifolds (shortly *e-Kähler*) in dependence on the smoothness class of the metric.

We present well known facts, which were proved by Domashev, Kurbatova, Mikeš, Prvanović, Otsuki, Tashiro etc.
In these results no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects.

Definition

An n-dimensional (pseudo-) Riemannian manifold (M, g) is called an *e-Kähler manifold K_n*, if beside the metric tensor g, a tensor field F ($\neq Id$) of type $(1, 1)$ is given on the manifold M_n, called a ***structure*** F, such that the following conditions hold:

$$(1) \qquad F^2 = e\, Id; \quad g(X, FX) = 0; \quad \nabla F = 0,$$

where $e = \pm 1$, X is an arbitrary vector of TM_n, and ∇ denotes the covariant derivative in K_n.

- If $e = -1$, K_n is a (*pseudo-*) *Kähler space* (also *elliptic Kähler space*) and F is a *complex structure*.

  As *A-spaces*, these spaces were first considered by P.A. Shirokov. Independently they were studied by E. Kähler .

- If $e = +1$, K_n is a *hyperbolic Kähler space* (also *para Kähler space*) and F is a *product structure*. The spaces K_n^+ were considered by P.K. Rashevskij.

The e-Kähler spaces introduced here are called shortly “Kähler”. By our definition we want to give a unified notation for all clases.

3. Holomorphically projective mapping theory for $K_n \to \bar{K}_n$ of class C^1

Assume the e-Kähler manifolds $K_n = (M, g, F)$ and $\bar{K}_n = (\bar{M}, \bar{g}, \bar{F})$ with metrics g and $\bar{g}$, structures F and $\bar{F}$, Levi-Civita connections ∇ and $\bar{\nabla}$, respectively.

Here K_n, $\bar{K}_n \in C^1$, i.e. $g, \bar{g} \in C^1$ which means that their components g_{ij}, $\bar{g}_{ij} \in C^1$.

Definition

A curve ℓ in K_n which is given by the equation $\ell = \ell(t)$, $\lambda = d\ell/dt$ $(\neq 0)$, $t \in I$, where t is a parameter is called *analytically planar*, if under the parallel translation along the curve, the tangent vector λ belongs to the two-dimensional distribution $D = Span\{\lambda, F\lambda\}$ generated by λ and its conjugate $F\lambda$, that is, it satisfies

$$\nabla_t \lambda = a(t)\lambda + b(t)F\lambda,$$

where $a(t)$ and $b(t)$ are some functions of the parameter t.

- Particularly, in the case $b(t) = 0$, an analytically planar curve is *a geodesic*.

Definition

A diffeomorphism $f\colon K_n \to \bar{K}_n$ is called a *holomorphically projective mapping* of K_n onto $\bar{K}_n$ if f maps any analytically planar curve in K_n onto an analytically planar curve in $\bar{K}_n$.

Assume a holomorphically projective mapping $f\colon K_n \to \bar{K}_n$.
Since f is a diffeomorphism, we can suppose local coordinate charts on M or $\bar{M}$, respectively, such that locally, $f\colon K_n \to \bar{K}_n$ maps points onto points with the same coordinates, and $\bar{M} = M$.

A manifold K_n admits a holomorphically projective mapping onto $\bar{K}_n$ if and only if the following equations:
(2)

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X + e\psi(FX)FY + e\psi(FY)FX$$

hold for any tangent fields X, Y and where ψ is a differential form.
• If $\psi \equiv 0$ than f is *affine* or *trivially holomorphically projective*.
Beside these facts it was proved that $\bar{F} = \pm F$; for this reason we can suppose that $\bar{F} = F$. In local form:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i + e\psi_{\bar{i}} \delta^h_{\bar{j}} + e\psi_{\bar{j}} \delta^h_{\bar{i}},$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of K_n and $\bar{K}_n$, ψ_i, F^h_i are components of ψ, F and δ^h_i is the Kronecker delta, $\psi_{\bar{i}} = \psi_\alpha F^\alpha_i$, $\delta^h_{\bar{i}} = F^h_i$.
Here and in the following we will use the conjugation operation of indices in the way

$$A_{\dots \bar{i} \dots} = A_{\dots k \dots} F^k_i .$$

Equations (2) are equivalent to the following equations
(3)
$$\nabla_Z \bar{g}(X,Y) = 2\psi(Z)\bar{g}(X,Y) + \psi(X)\bar{g}(Y,Z) + \psi(Y)\bar{g}(X,Z)$$
$$-e\psi(FX)\bar{g}(FY,Z) - e\psi(FY)\bar{g}(FX,Z).$$

In local form:

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} - e\psi_{\bar{i}}\bar{g}_{\bar{j}k} - e\psi_{\bar{j}}\bar{g}_{\bar{i}k},$$

where " , " denotes the covariant derivative on K_n. It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+2)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad \partial_i = \partial/\partial x^i.$$

Domashev, Kurbatova and Mikeš proved that equations (2) and (3) are equivalent to

$$(4)\qquad \begin{gathered} \nabla_Z a(X,Y) = \lambda(X)g(Y,Z) + \lambda(Y)g(X,Z) - \\ e\lambda(FX)g(FY,Z) - e\lambda(FY)g(FX,Z). \end{gathered}$$

In local form:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{i}} g_{\bar{j}k} - e\lambda_{\bar{j}} g_{\bar{i}k},$$

where

$$(5)\quad \text{(a)}\ \ a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \text{(b)}\ \ \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha.$$

From (4) follows $\lambda_i = \partial_i \lambda = \partial_i(\frac{1}{4}\, a_{\alpha\beta} g^{\alpha\beta})$. On the other hand:

$$(6)\quad \bar{g}_{ij} = \mathrm{e}^{2\Psi} \tilde{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \tilde{g}}{\det g} \right|, \quad \|\tilde{g}_{ij}\| = \|g^{i\alpha} g^{j\beta} a_{\alpha\beta}\|^{-1}.$$

The above formulas are the criterion for holomorphically projective mappings $K_n \to \bar{K}_n$, globally as well as locally.

4. Holomorphically projective mapping theory for $K_n \to \bar{K}_n$ of class C^2

Let K_n and $\bar{K}_n \in C^2$ be e-Kähler manifolds, then for holomorphically projective mappings $K_n \to \bar{K}_n$ the Riemann and the Ricci tensors transform in this way

(7)

(a) $\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} - e\delta^h_{\bar{k}} \psi_{i\bar{j}} + e\delta^h_{\bar{j}} \psi_{i\bar{k}} + 2e\delta^h_{\bar{i}} \psi_{j\bar{k}};$

(b) $\bar{R}_{ij} = R_{ij} - (n+2)\psi_{ij},$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j + \psi_{\bar{i}} \psi_{\bar{j}}$ $(\psi_{ij} = \psi_{ji} = -e\psi_{\bar{i}\bar{j}})$.
The tensor of holomorphically projective curvature, which is defined in the following form

(8)

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left(\delta^h_k R_{ij} - \delta^h_j R_{ik} - e\delta^h_{\bar{k}} R_{i\bar{j}} + e\delta^h_{\bar{j}} R_{i\bar{k}} + 2e\delta^h_{\bar{i}} R_{j\bar{k}}\right),$$

is invariant with respect to holomorphically projective mappings, i.e. $\bar{P}^h_{ijk} = P^h_{ijk}$.

of equations (4)

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{i}} g_{\bar{j}k} - e\lambda_{\bar{j}} g_{\bar{i}k},$$

where

$$\text{(a)} \quad a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \text{(b)} \quad \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha .$$

have the following form

(9)

$$\begin{aligned} a_{i\alpha} R^{\alpha}_{jkl} + a_{j\alpha} R^{\alpha}_{ikl} \quad = \quad & g_{ik}\lambda_{j,l} + g_{jk}\lambda_{i,l} - g_{il}\lambda_{j,k} - g_{jl}\lambda_{i,k} \\ & - e g_{\bar{i}k}\lambda_{\bar{j},l} - e g_{\bar{j}k}\lambda_{\bar{i},l} + e g_{\bar{i}l}\lambda_{\bar{j},k} + e g_{\bar{j}l}\lambda_{\bar{i},k} . \end{aligned}$$

We make the remark that the formulas introduced above, (7), (8) and (9), are not valid when $K_n \notin C^2$ or $\bar{K}_n \notin C^2$.

After contraction with g^{jk} we get:

$$a_{i\alpha}R^{\alpha}_{k} + a_{\alpha\beta}R^{\alpha\beta}_{ik} = \mu g_{ik} + e\lambda_{\bar{i},\bar{k}} - (n-1)\lambda_{i,k},$$

where $R^{\alpha}{}_{il}{}^{\beta} = g^{\beta k}R^{\alpha}{}_{ilk}$; $R^{\alpha}_{l} = g^{\alpha j}R_{jl}$ and $\mu = \lambda_{\alpha,\beta}g^{\alpha\beta}$.
We contract this formula with $F^{i}_{i'}F^{k}_{k'}$ and from the properties of the Riemann and the Ricci tensors of K_n we obtain

(10) $$\lambda_{\bar{i},\bar{k}} = -e\lambda_{i,k},$$

and

(11) $$n\lambda_{i,k} = \mu g_{ik} - a_{i\alpha}R^{\alpha}_{k} - a_{\alpha\beta}R^{\alpha}{}_{ik}{}^{\beta}\ .$$

Because λ_i is a gradient-like covector, from equation (11) follows $a_{i\alpha}R^{\alpha}_{j} = a_{j\alpha}R^{\alpha}_{i}$.
From (10) follows that the vector field $\lambda_{\bar{i}}$ $(\equiv \lambda_{\alpha}F^{\alpha}_{i})$ is a Killing vector field, i.e. $\lambda_{\bar{i},j} + \lambda_{\bar{j},i} = 0$.

5. Holomorphically projective mappings between $K_n \in C^r$ $(r > 2)$ and $\bar{K}_n \in C^2$

We proof the following theorem

Theorem

If $K_n \in C^r$ $(r > 2)$ admits holomorphically projective mappings onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^r$.

The proof of this theorem follows from the following lemmas.

Lemma 1

Let $\lambda^h \in C^1$ be a vector field and ϱ a function. If

$$\partial_i \lambda^h - \varrho\, \delta_i^h \in C^1 \tag{12}$$

then $\lambda^h \in C^2$ and $\varrho \in C^1$.

In a similar way we can prove the following: *if $\lambda^h \in C^r$ $(r \geq 1)$ and $\partial_i \lambda^h - \varrho \delta_i^h \in C^r$ then $\lambda^h \in C^{r+1}$ and $\varrho \in C^r$.*

Lemma 2

If $K_n \in C^3$ admits a holomorphically projective mapping onto $\bar{K}_n \in C^2$, then $\bar{K}_n \in C^3$.

Sketch of the proof:

In this case equations (4) and (11) hold. According to the assumptions $g_{ij} \in C^3$ and $\bar{g}_{ij} \in C^2$. By a simple check-up we find $\Psi \in C^2$, $\psi_i \in C^1$, $a_{ij} \in C^2$, $\lambda_i \in C^1$ and $R^h_{ijk}, R^h{}_{ij}{}^k, R_{ij}, R^h_i \in C^1$. From the above-mentioned conditions we easily convince ourselves that we can write equation (11) in the form (12), where

$$\lambda^h = g^{h\alpha}\lambda_\alpha \in C^1,\ \varrho = \mu/n \text{ and}$$

$$f_i^h = \frac{1}{n}\left(-\lambda^\alpha \Gamma^h_{\alpha i} - g^{h\gamma} a_{\alpha\gamma} R^\alpha_i + g^{h\gamma} a_{\alpha\beta} R^\alpha{}_{i\gamma}{}^\beta\right) \in C^1.$$

From Lemma 1 follows that $\lambda^h \in C^2$, $\varrho \in C^1$, and evidently $\lambda_i \in C^2$. Differentiating (4) twice we convince ourselves that $a_{ij} \in C^3$. From this and formula (6) follows that also $\Psi \in C^3$ and $\bar{g}_{ij} \in C^3$.

For holomorphically projective mappings between e-Kähler manifolds K_n and $\bar{K}_n$ of class C^3 holds the following third set of equations:

$$\mu_{,k} = -2\lambda_\alpha R^\alpha_k. \qquad (13)$$

- If $K_n \in C^r$ and $\bar{K}_n \in C^2$, then by Lemma 2, $\bar{K}_n \in C^3$ and (13) holds.

Because the system (4)

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{i}} g_{\bar{j}k} - e\lambda_{\bar{j}} g_{\bar{i}k},$$

where (a) $a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}$; (b) $\lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha$,

(11) $n\lambda_{i,k} = \mu g_{ik} + a_{i\alpha} R^\alpha_k + a_{\alpha\beta} R^\alpha{}_{ik}{}^\beta$

and (13)

$$\mu_{,k} = -2\lambda_\alpha R^\alpha_k$$

is closed, we can differentiate equations (4) $(r-1)$ times. So we convince ourselves that $a_{ij} \in C^r$, and also $\bar{g}_{ij} \in C^r$ $(\equiv \bar{K}_n \in C^r)$.

Remark

Moreover, in this case from equation (13) follows that the function $\mu \in C^{r-1}$.

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Thank you for your attention!