On a Class of Linear Weingarten Surfaces

Vladimir Pulov[1]
Mariana Hadzhilazova,[2] Ivailo Mladenov[2]

[1]Department of Physics, Technical University of Varna

[2]Institute of Biophysics, Bulgarian Academy of Science

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Outline

A Linear Weingarten Surface of Revolution

1. Some Differential Geometry
   - Principal Curvatures (k_μ, k_π)
   - First and Second Fundamental Forms
2. Linear Weingarten Surfaces (LW-surfaces)
   - $k_\mu = ck_\pi, \ c = \text{const}$
3. A LW-surface of Revolution: $k_\mu = 3k_\pi$
   - Monge and Whewell Parameterizations
   - Parameterizations via Elliptic Functions
   - Isothermal and Arclength Related Parameterizations
4. Geometrical Applications

Surface of Revolution

If $\mathcal{S}$ is a surface of revolution given by

$$\mathbf{x}(u,v) = (h(u)\cos v, h(u)\sin v, g(u))$$

then its principal curvatures can be found by

$$k_\mu = \frac{g''h' - g'h''}{(g'^2 + h'^2)^{3/2}}, \qquad k_\pi = \frac{g'}{h(g'^2 + h'^2)^{1/2}}$$

where $g' \equiv \mathrm{d}g/\mathrm{d}u$, etc.

Surface of Revolution

If $\mathcal{S}$ is a surface of revolution given by

$$\mathbf{x}(u,v) = (h(u)\cos v, h(u)\sin v, g(u))$$

then the coefficients of its I (FFF) and II (SFF)

$$I = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2, \qquad II = L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2$$

are calculated by

$$E = h'^2 + g'^2, \qquad F = 0, \qquad G = h^2$$

$$L = (h''g' - h'g'')/\sqrt{E}, \qquad M = 0, \qquad N = hg'/\sqrt{E}$$

where $g' \equiv \mathrm{d}g/\mathrm{d}u$, etc.

The Shape of a Rotating Liquid Drop (Mladenov and Oprea, 2016)

- Incompressible fluid body under surface tension is rotating with constant angular velocity.
- The fluid surface is in a state of equilibria, effectively described by the mean curvature of the form

$$H = 2\tilde{a}r^2 + \tilde{c}, \quad \tilde{a} > 0, \quad \tilde{c} = \text{const}, \quad r - \text{radius}$$

- The principal curvatures of such surface of revolution obey a linear relation

$$k_\mu = 3k_\pi - 2\tilde{c}$$

which makes the rotating drop a linear Weingarten surface.

Linear Weingarten Surfaces

Surfaces whose principal curvatures obey a linear relation

$$k_\mu = ck_\pi, \qquad c = n + 1, \qquad n = 0, 1, 2, \ldots$$

are referred to as LW(n)-surfaces.

LW(n)-Surfaces of Revolution

- LW(0) Sphere ($k_\mu = k_\pi$)
- LW(1) Mylar Balloon ($k_\mu = 2k_\pi$)
- LW(2)-Balloon ($k_\mu = 3k_\pi$)

Variational Characterization of LW(n)-Balloons (Mladenov and Oprea, 2007)

Find the profile curve $z = z(u), \quad z(r) = 0, \quad r > 0$

of a surface of revolution $\mathbf{x}(u, v) = (u\cos v, u\sin v, z(u))$

by extremizing the n^{th} moment $J_n(z) = \int_0^r u^n z(u)\mathrm{d}u, \ n = 0, 1, \ldots$

subject to the constraint $\int_0^r \sqrt{1 + z'(u)^2}\,\mathrm{d}u = a > 0 \text{ (fixed)}$

and the transversality condition $\lim_{u \to r^-} z'(u) = -\infty$

Variational Characterization of LW(n)-Balloons (Mladenov and Oprea, 2007)

The surface $\mathcal{S}$ that solves the n^{th}-moment variational problem is LW(n)-balloon with the profile curve parameterized by

$$z(u) = \frac{r}{2(n+1)}\left[B_1\left(\frac{n+2}{2(n+1)}, \frac{1}{2}\right) - B_t\left(\frac{n+2}{2(n+1)}, \frac{1}{2}\right)\right]$$

$$t = \left(\frac{u}{r}\right)^{2(n+1)}, \qquad u \in [0, r], \qquad n = 0, 1, 2, \ldots$$

where $B_t(p, q)$ denotes the incomplete Beta function of the real variable t and the parameters p and q.

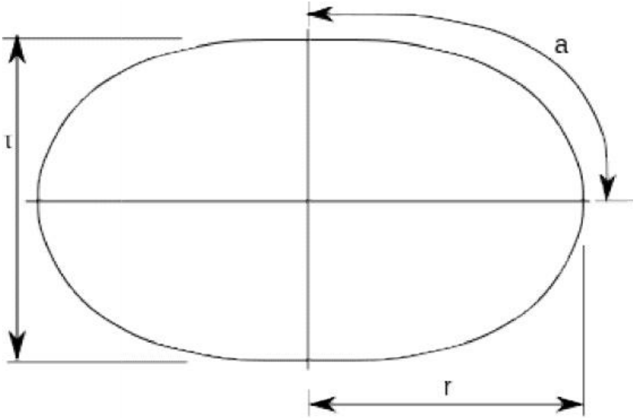
Characterizations of LW(n)-Balloons

- Profile arclength is fixed

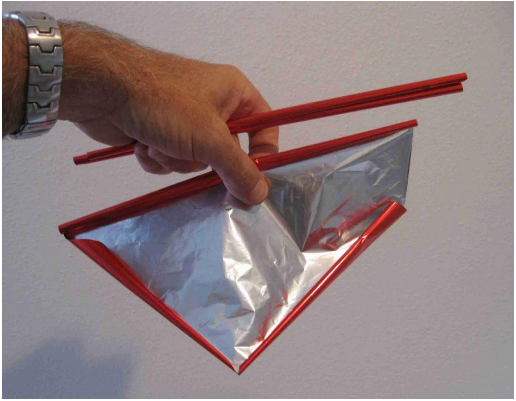
$$\int_0^r \sqrt{1 + z'(u)^2}\, \mathrm{d}u = a > 0$$

- LW(0) (Sphere) is a surface with maximum area $J_0(z)$ of the meridional section for a given profile arclength.
- LW(1) (Mylar) is a surface with maximum volume $J_1(z)$ for a given profile arclength.
- LW(2)-Balloon $(k_\mu = 3k_\pi)$ is a surface with extremal second moment $J_2(z)$ for a given profile arclength.

Profile of LW(n)-Balloon



Sheets of Mylar Polyester Foil
(a material used for construction of the Mylar balloon)



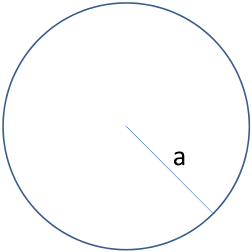
Mylar is a Trademark

- Mylar is extremely thin polyester film.
- Mylar is flexible and inelastic material.
- Mylar is having a great tensile stress.

Constructing the Mylar Balloon

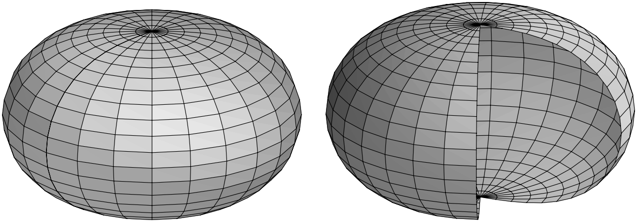
- Take two circular disks made of Mylar.
- Sew the disks together along their boundaries.
- Inflate with either air or helium.

The Deflated Mylar Balloon
(two circular disks made of Mylar foil sewn together)



The Inflated Mylar Balloon
(physical prototype of the Mylar balloon)

The Mylar Balloon via Mathematica®
(it resembles a slightly flattened sphere)



First Geometrical Depiction
(Paulsen, 1994)

- Palsen paused the problem in variational settings. He observed that the corresponding Euler-Lagrange equation has no closed form solution in elementary functions.
- Palsen succeeded to determine the radius, the thickness and the volume of the inflated Mylar balloon in terms of Gamma function.

The Profile of the Mylar Balloon via Elliptic Integrals and Jacobian Elliptic Functions (Mladenov and Oprea, 2003)

$$x(u) = r\,\mathrm{cn}(u,\,k)$$

$$z(u) = r\sqrt{2}\left[E(\mathrm{sn}(u,\,k),\,k) - \frac{1}{2}F(\mathrm{sn}(u,\,k),\,k)\right]$$

$$k = \frac{1}{\sqrt{2}}, \qquad u \in [-K(k),\,K(k)]$$

Here $F(\cdot, k)$ and $E(\cdot, k)$ are the incomplete elliptic integrals of first and second order, $K(k)$ is the complete elliptic integrals of first order, $\mathrm{sn}(\cdot, k)$ and $\mathrm{cn}(\cdot, k)$ are the Jacobian elliptic functions all with modulus k.

The Mylar Balloon via Jacobian Elliptic Functions (Mladenov and Oprea, 2003)

$$E = \frac{r^2}{2} \qquad F = 0 \qquad G = r^2 \operatorname{cn}^2\left(u, \frac{1}{\sqrt{2}}\right)$$

$$L = r \operatorname{cn}\left(u, \frac{1}{\sqrt{2}}\right) \qquad M = 0 \qquad N = r \operatorname{cn}^3\left(u, \frac{1}{\sqrt{2}}\right)$$

The Mylar Balloon in Conformal Representation (Mladenov, 2004)

$$x(u) = \frac{r}{\sqrt{\cosh(2\,u)}}\cos v, \qquad y(u) = \frac{r}{\sqrt{\cosh(2\,u)}}\sin v$$

$$z(u) = r\sqrt{2}\left[E(\varphi,\ k) - \frac{1}{2}F(\varphi,\ k)\right]$$

$$\varphi = \frac{\sqrt{2}\,\sinh(u)}{\sqrt{\cosh(2\,u)}}, \quad k = \frac{1}{\sqrt{2}}, \quad u \in (-\infty, \infty), \quad v \in [0, 2\pi]$$

Here $F(\varphi, k)$ and $E(\varphi, k)$ are the incomplete elliptic integrals of first and second order.

The Mylar Balloon in Conformal Representation
(Mladenov, 2004)

$$E = \frac{r^2}{\cosh(2u)} \qquad F = 0 \qquad G = \frac{r^2}{\cosh(2u)}$$

$$L = \frac{2r}{\cosh^{3/2}(2u)} \qquad M = 0 \qquad N = \frac{2r}{\cosh^{3/2}(2u)}$$

The Mylar Profile in Whewell Representation (Hadzhilazova and Mladenov, 2008)

$$x(\theta) = r\sqrt{\sin\theta}$$

$$z(\theta) = r\left(\frac{1}{k}E(\arccos(\sqrt{\sin\theta}), k) - kF(\arccos(\sqrt{\sin\theta}), k)\right)$$

$$k = \frac{1}{\sqrt{2}}, \quad r > 0, \quad \theta \in [0, \pi]$$

r – radius of the balloon

Here $F(\cdot, k)$ and $E(\cdot, k)$ are the incomplete elliptic integrals of first and second order with modulus $k = 1/\sqrt{2}$.

The Mylar Profile via the Weierstrassian Functions
(Pulov, Hadzhilazova and Mladenov, 2015)

$$x(u) = r\,\frac{2\wp(u) - r^2}{2\wp(u) + r^2}$$

$$z(u) = 2\zeta(u) + \frac{2\wp'(u)}{2\wp(u) + r^2}, \quad u \in \left[-\frac{\omega}{2}, \frac{\omega}{2}\right], \quad \omega = \frac{\tilde{\omega}}{r}$$

Here $\wp(u)$, $\wp'(u)$ and $\zeta(u)$ are the Weierstrassian $\wp$-function, its derivative $\wp'(u)$ and the Weierstrassian zeta function built with the invariants $g_2 = -r^4$ and $g_3 = 0$; $r > 0$ is the radius of the balloon; $\tilde{\omega} \approx 2.6220$ is the lemniscate constant.

The Mylar Balloon via the Weierstrassian Functions
(Pulov, Hadzhilazova and Mladenov, 2015)

$$E = r^4 \qquad F = 0 \qquad G = r^2 \left(\frac{2\wp(u) - r^2}{2\wp(u) + r^2} \right)^2$$

$$L = 2r^3 \left(\frac{2\wp(u) - r^2}{2\wp(u) + r^2} \right) \qquad M = 0 \qquad N = r \left(\frac{2\wp(u) - r^2}{2\wp(u) + r^2} \right)^3$$

Namihaya Sports Hall Dome, Kadoma City, Japan
(designed in the form of Mylar balloon, 1996)

LW(2)-Balloon $(k_\mu = 3k_\pi)$
Monge Parameterization

The profile curve $\gamma(x) = (x, 0, z(x))$ of LW(2) is represented by

$$z(x) = r\left(\frac{\sqrt{3}-1}{2\sqrt[4]{3}}F(\varphi,k) - \sqrt[4]{3}\,E(\varphi,k) + \frac{\sqrt{1-(x/r)^6}}{\sqrt{3}+1-(x/r)^2}\right)$$

$$\varphi = \arccos\frac{\sqrt{3}-1+(x/r)^2}{\sqrt{3}+1-(x/r)^2}, \quad k = \sqrt{\frac{2+\sqrt{3}}{4}}, \quad x \in [0, r]$$

where $F(\varphi, k)$ and $E(\varphi, k)$ are the incomplete elliptic integrals of first and second order with modulus k and $r > 0$ is the radius of the balloon.

Inversion of Elliptic Integrals

The Jacobian elliptic amplitude function

$$\varphi = \mathrm{am}\; u_1$$

is defined as the inverse of the first kind elliptic integral

$$F(\varphi, k) = u_1 = \int_0^{\varphi} \frac{\mathrm{d}t}{\sqrt{1 - k^2 \sin^2 t}}$$

LW(2)-Balloon $(k_\mu = 3k_\pi)$
The Profile Curve Parameterized via $\varphi = \mathrm{am}\, u_1$

$$x(\varphi) = r\sqrt{\frac{1-\sqrt{3}+(1+\sqrt{3})\cos\varphi}{1+\cos\varphi}}$$

$$z(\varphi) = r\left(\frac{1-\sqrt{3}}{2\sqrt[4]{3}}F(\varphi,k) + \sqrt[4]{3}\,E(\varphi,k) - \frac{\sqrt[4]{3}\sin\varphi\sqrt{1-k^2\sin^2\varphi}}{1+\cos\varphi}\right)$$

$$\varphi \in \left[-\arccos\frac{\sqrt{3}-1}{\sqrt{3}+1},\ \arccos\frac{\sqrt{3}-1}{\sqrt{3}+1}\right], \quad k = \sqrt{\frac{2+\sqrt{3}}{4}}, \quad r > 0$$

Here $F(\varphi,k)$ and $E(\varphi,k)$ are the incomplete elliptic integrals of first and second order with modulus k and $r > 0$ is the radius of the balloon.

LW(2)-Balloon via $\varphi = \operatorname{am} u_1$

First Fundamental Form

$$E = \frac{4r^2\cos^2\left(\frac{\varphi}{2}\right)}{\sqrt{3}\left((1+\sqrt{3})\cos(\varphi)-\sqrt{3}+1\right)\left((2+\sqrt{3})\cos(2\varphi)-\sqrt{3}+6\right)}$$

$$F = 0$$

$$G = \frac{r^2\left((1+\sqrt{3})\cos(\varphi)-\sqrt{3}+1\right)}{\cos(\varphi)+1}$$

LW(2)-Balloon ($k_\mu = 3k_\pi$)
Arclength Related Parameterization

The profile curve of LW(2) is represented by

$$\begin{aligned} x(u) &= 2r\sqrt{-\wp(u)} \\ z(u) &= 2r\,\zeta(u) + \alpha(r), \qquad u \in [0, 2\pi) \end{aligned}$$

where $\wp(u)$ and $\zeta(u)$ are the Weierstrassian $\wp$ and zeta functions built with the invariants $g_2 = 0$ and $g_3 = -1/16$; $\alpha(1) = -1.29$; $r > 0$ is the radius of the balloon.

The parameter u and the arclength parameter s are related by

$$s = \frac{r}{4}\int \frac{\mathrm{d}u}{\sqrt{-\wp(u)}}$$

Arclength Related Parameterization

First and Second Fundamental Forms

$$FFF = \left\{ -\frac{r^2}{16\wp(u)},\ 0,\ -4r^2\wp(u) \right\}$$

$$SFF = \left\{ \frac{3r}{4},\ 0,\ 16r\wp^2(u) \right\}$$

LW(2)-Balloon $(k_\mu = 3k_\pi)$
The Profile Curve in Whewell Parameterization

$$x(\theta) = r\sqrt[3]{\sin\theta}$$

$$z(\theta) = r\left(\frac{\sqrt{3}-1}{2\sqrt[4]{3}}F(\varphi,k) - \sqrt[4]{3}\,E(\varphi,k) + \frac{\cos\theta}{1+\sqrt{3}-\sqrt[3]{\sin^2\theta}}\right)$$

$$\varphi = \arccos\frac{\sqrt{3}-1+\sqrt[3]{\sin^2\theta}}{\sqrt{3}+1-\sqrt[3]{\sin^2\theta}}, \quad k = \sqrt{\frac{2+\sqrt{3}}{4}}, \quad \theta \in \left[0, \frac{\pi}{2}\right]$$

Here $F(\varphi,k)$ and $E(\varphi,k)$ are the incomplete elliptic integrals of first and second order with modulus k and $r > 0$ is the radius of the balloon.

LW(2)-Balloon ($k_\mu = 3k_\pi$)
An Alternative Whewell Parameterization

The profile curve $\gamma(\theta) = (\, x(\theta),\, 0,\, z(\theta)\,)$ of LW(2) is represented by

$$x(\theta) = r\,\sqrt[3]{\sin\theta}$$

$$z(\theta) = -\frac{r\cos\theta}{3}\;{}_2F_1\Big(\frac{1}{2},\,\frac{1}{3};\,\frac{3}{2};\,\cos^2\theta\Big),\qquad \theta\in[0,\pi]$$

where ${}_2F_1(\frac{1}{2},\,\frac{1}{3};\,\frac{3}{2};\,\zeta)$ is the Gauss' hypergeometric function and $r > 0$ is the radius of the balloon.

An Alternative Whewell Parameterization
First and Second Fundamental Forms

$$FFF = \left\{ \frac{r^2}{9\sin^{4/3}\theta},\quad 0,\quad r^2 \sin^{2/3}\theta \right\}$$

$$SFF = \left\{ -\frac{r}{3\sin^{2/3}\theta},\quad 0,\quad -r\sin^{4/3}\theta \right\}$$

LW(2)-Balloon $(k_\mu = 3k_\pi)$
Parameterization via Isothermal Coordinates

By imposing the relation $\sin\theta\cosh 3u = 1$ the profile curve of LW(2) takes the form

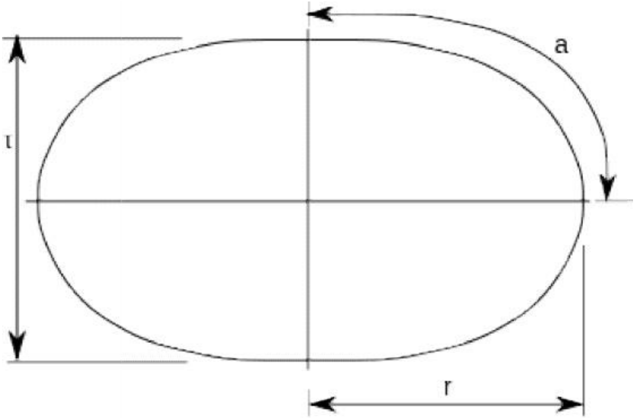
$$x(u) = \frac{r}{\sqrt[3]{\cosh(3u)}}, \quad z(u) = r\int_0^u \frac{\mathrm{d}t}{\cosh(3t)\sqrt[3]{\cosh(3t)}}$$

with the I and the II fundamental forms of LW(2) expressed by

$$FFF = \left\{\frac{r^2}{\cosh^{\frac{2}{3}}(3u)},\ 0,\ \frac{r^2}{\cosh^{\frac{2}{3}}(3u)}\right\}$$

$$SFF = \left\{\frac{3r}{\cosh^{\frac{4}{3}}(3u)},\ 0,\ \frac{r}{\cosh^{\frac{4}{3}}(3u)}\right\}.$$

Profile of LW(n)-Balloon



Exact Expressions

Quantity	Sphere	Mylar	$LW(2)$-Balloon
a/r	$\frac{\pi}{2}$	$\frac{\tilde{\omega}}{2}$	$\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)}{6\,\Gamma\left(\frac{2}{3}\right)}$
$\tau/(2r)$	1	$\frac{\pi}{2\tilde{\omega}}$	$\frac{\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)}{6\,\Gamma\left(\frac{7}{6}\right)}$

r – radius, τ – thickness, a – 1/4 profile arclength

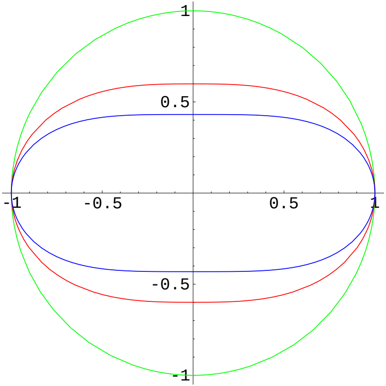
$$\tilde{\omega} = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}, \qquad \tilde{\omega} \approx 2.6220 \text{ – lemniscate constant}$$

Approximate Values

Quantity	Sphere	Mylar	$LW(2)$-Balloon
a/r	1.5708	1.3110	1.2143
$\tau/(2r)$	1	0.5991	0.4312

r – radius, τ – thickness, a – 1/4 profile arclength

Profiles of Sphere, Mylar and LW(2)-Balloon
(with the same diameters)



Exact Expressions

Quantity	Sphere	Mylar	$LW(2)$-Balloon
Σ	πr^2	$2r^2$	$\dfrac{2\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)}{3\Gamma\left(\frac{4}{3}\right)} r^2$
A	$4\pi r^2$	$\pi^2 r^2$	$\dfrac{2\pi^{3/2}\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{5}{6}\right)} r^2$
V	$\frac{4}{3}\pi r^3 \; (= V_0)$	$\frac{1}{3}\pi\tilde{\omega} r^3$	$\frac{2}{3}\pi r^3 \; \left(= \frac{V_0}{2}\right)$

Σ – cross section area, A – surface area, V – volume

$\tilde{\omega} \approx 2.6220$, $\Gamma(\cdot)$ – Gamma function

Approximate Values

Quantity	Sphere	Mylar	$LW(2)$-Balloon
Σ/r^2	3.1416	2	1.4937
A/r^2	12.5664	9.8696	8.8102
V/r^3	4.1888	2.7457	2.0944

Σ – cross section area, A – surface area, V – volume

Sphericity Index

- Sphericity is the measure of how closely the shape of an object approaches that of a mathematically perfect object.
- For example, the sphericity of the balls inside a ball bearing determines the quality of the bearing, such as the load it can bear or the speed at which it can turn without failing.
- Defined by Hakon Wadell in 1935, sphericity is a specific example of a compactness measure of a shape.
- The sphericity of a particle is the ratio of the surface area of a sphere (of the same volume V of the given particle) to the surface area A of the particle.

$$\text{Sphericity} = \frac{\sqrt[3]{36\pi}\, V^{2/3}}{A}$$

Exact Expressions

Quantity	Sphere	Mylar	$LW(2)$-Balloon
Sphericity	1	$\dfrac{(2\tilde{\omega})^{2/3}}{\pi}$	$\dfrac{3\sqrt[3]{2}\,\Gamma\left(\frac{5}{6}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}$
Homogeneity	1	$\dfrac{\pi^{3/2}}{2\tilde{\omega}}$	$\dfrac{\pi^{3/4}\left[\Gamma\left(\frac{1}{3}\right)\right]^{3/2}}{3\sqrt{6}\left[\Gamma\left(\frac{5}{6}\right)\right]^{3/2}}$

$$\text{Sphericity} = \frac{\sqrt[3]{36\pi}\,V^{2/3}}{A}, \quad \text{Homogeneity} = \frac{A^{3/2}}{6\sqrt{\pi}\,V}$$

Approximate Values

Quantity	Sphere	Mylar	$LW(2)$-Balloon
Sphericity	1	0.9608	0.8985
Homogeneity	1	1.0618	1.1741

$$\text{Sphericity} = \frac{\sqrt[3]{36\pi}\, V^{2/3}}{A}, \quad \text{Homogeneity} = \frac{A^{3/2}}{6\sqrt{\pi}\, V}$$

Bolshoy Ice Dome, Sochi, Russia, 2012

Looks like LW(2)-balloon, isn't it?

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