Fisher metric for diagonalizable quadratic Hamiltonians and application to phase transitions

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Overview

1. Thermo Field Dynamics and entanglement entropy
   - Statistical mechanics of many-particle systems
   - Thermo field dynamics (TFD)
   - Extended entanglement entropy

2. Riemannian metric on the space of parameters
   - Fisher information metric
   - Example: two fourth-order PU oscillators
   - Metric invariants and application to phase transitions

Statistical treatment of a large number of interacting particles

- Statistical average:

The essential quantity in statistical mechanics in thermal equilibrium is the statistical average of a quantity A, say over the grand canonical ensemble at temperature T given by

$$\langle A \rangle = Z^{-1}(\beta)\,\mathrm{Tr}[A\,e^{-\beta\,H}]. \tag{1}$$

- The partition function:

$$Z(\beta) = \mathrm{Tr} e^{-\beta\,H}. \tag{2}$$

- The inverse temperature:

$$\beta = k_B\,T^{-1}. \tag{3}$$

Basics of thermo field dynamics (TFD)

- Matsubara observation (1955) [T. Matsubara, Prog. Theor. Phys., 14, 351, 1955]:

The statistical average $\langle A \rangle$ has the properties similar to the vacuum expectation value of A in quantum field theory!

- The TFD formalism (1975) [Y. Takahashi, H. Umezawa, Collective Phenomena 2, 55, 1975]:

Construct a field theory in which the vacuum expectation value coincides with the statistical average, i.e.

$$\langle A \rangle = Z^{-1}(\beta)\, \mathrm{Tr}[A\, e^{-\beta H}] \equiv \langle 0(\beta)|A|0(\beta)\rangle . \tag{4}$$

- Here $|0(\beta)\rangle$ is the temperature dependent vacuum state in a new space to be constructed.

Basics of thermo field dynamics (TFD)

- General considerations:

Define a suitable thermal state $|0(\beta)\rangle$ which satisfy

$$\langle 0(\beta)|F|0(\beta)\rangle = Z^{-1}(\beta) \sum_n \langle n|\widehat{F}|n\rangle \, e^{-\beta E_n}, \tag{5}$$

for arbitrary dynamical variable F, where

$$H|n\rangle = E_n|n\rangle , \quad \langle n|m\rangle = \delta_{nm}. \tag{6}$$

Now expand the thermal state $|0(\beta)\rangle$ in terms of the energy eigenstates $|n\rangle$:

$$|0(\beta)\rangle = \sum_n |n\rangle \, f_n(\beta). \tag{7}$$

Insert (7) back in (5) to get

$$f_n^*(\beta)\, f_m(\beta) = Z^{-1}(\beta)\, e^{-\beta E_n}\, \delta_{mn}. \tag{8}$$

Basics of thermo field dynamics (TFD)

Equation (8) cannot be satisfied for mere numbers $f_n(\beta)$, but one notices that it can be regarded as the orthogonality condition in a Hilbert space in which the expansion coefficient $f_n(\beta)$ is a vector. In other words,

the state $|0(\beta)\rangle$ is a vector in the space spanned by $|n\rangle$ and $f_n(\beta)$.

- Adding fictitious degrees of freedom.

In order to realize such a representation we introduce a fictitious system which is of exactly the same structure as physical one under consideration.

The new tilde system is described by the Hamiltonian $\tilde{H}$ and the tilde Hilbert space is spanned by the vectors $|\tilde{n}\rangle$:

$$\tilde{H}\,|\tilde{n}\rangle = E_n\,|\tilde{n}\rangle\,, \quad \langle\tilde{n}|\tilde{m}\rangle = \delta_{nm}. \tag{9}$$

The double Hilbert space

We then consider the space spanned by the direct product of $|n\rangle$ and $|\tilde{n}\rangle$ with properties:

$$\langle \tilde{m}, n|F|n', \tilde{m}'\rangle = \langle n|F|n'\rangle\, \delta_{mm'}\,, \tag{10}$$

$$\langle \tilde{m}, n|\tilde{F}|n', \tilde{m}'\rangle = \langle \tilde{m}|\tilde{F}|\tilde{m}'\rangle\, \delta_{nn'}\,, \tag{11}$$

$$\langle n|F|m\rangle = \langle \tilde{n}|\tilde{F}^{\dagger}|\tilde{m}\rangle. \tag{12}$$

If we now choose

$$f_n(\beta) = |\tilde{n}\rangle\, e^{-\beta\, E_n}\, Z^{-1/2}(\beta)\,, \tag{13}$$

the relation (8) is satisfied due to (9). With this one can obtain the thermal state $|0(\beta)\rangle$:

$$|0(\beta)\rangle = Z^{-1}(\beta) \sum\nolimits_n e^{-\beta\, E_n}\, |n, \tilde{n}\rangle. \tag{14}$$

Thermal equilibrium vacuum state for bosons and fermions

- The total Hamiltonian in the double Hilbert space is given by

$$\widehat{H} = H - \tilde{H}, \tag{15}$$

with $H = E_k\, a_k^\dagger\, a_k$ and $H = E_k\, \tilde{a}_k^\dagger\, \tilde{a}_k$, is invariant under fermionic thermal Bogoliubov transformations:

$$a_{k,\beta} = a_k\, \cosh\theta(k,\beta) - \tilde{a}_k^\dagger\, \sinh\theta(k,\beta), \tag{16}$$

$$\tilde{a}_{k,\beta} = \tilde{a}_k\, \cosh\theta(k,\beta) - a_k^\dagger\, \sinh\theta(k,\beta), \tag{17}$$

or bosonic thermal Bogoliubov transformations:

$$a_{k,\beta} = a_k\, \cos\theta(k,\beta) - \tilde{a}_k^\dagger\, \sin\theta(k,\beta), \tag{18}$$

$$\tilde{a}_{k,\beta} = \tilde{a}_k\, \cos\theta(k,\beta) - a_k^\dagger\, \sin\theta(k,\beta). \tag{19}$$

- The vacua are connected by

$$|0(\beta)\rangle = e^{-i\,G}\, |0,\tilde{0}\rangle, \tag{20}$$

with generator $G = i\,\sum_k \theta\,(a_k^\dagger\, \tilde{a}_k^\dagger - \tilde{a}_k\, a_k)$.

The TFD Fock space

- One can show that

$$|0(\beta)\rangle = \begin{cases} \prod_k \left(\cos\theta_k(\beta) + \sin\theta_k(\beta)\, a_k^\dagger\, \tilde{a}_k^\dagger\right) |0,\tilde{0}\rangle, & \textit{for fermions}, \\ \prod_k \frac{1}{\cosh\theta_k(\beta)}\, e^{\tanh\theta_k(\beta)\, a_k^\dagger\, \tilde{a}_k^\dagger} |0,\tilde{0}\rangle, & \textit{for bosons}. \end{cases} \tag{21}$$

- One particle state for bosons:

$$a^\dagger(\beta)|0(\beta)\rangle = \frac{1}{\sqrt{f_B(\omega)}}\,\tilde{a}\,|0(\beta)\rangle = \frac{1}{\sqrt{1+f_B(\omega)}}\,a^\dagger\,|0(\beta)\rangle, \tag{22}$$

- One particle state for fermions:

$$a^\dagger(\beta)|0(\beta)\rangle = -\frac{1}{\sqrt{f_F(\omega)}}\,\tilde{a}\,|0(\beta)\rangle = \frac{1}{\sqrt{1-f_B(\omega)}}\,a^\dagger\,|0(\beta)\rangle, \tag{23}$$

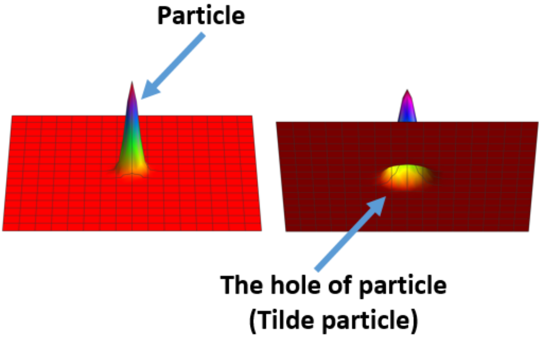
where

$$f_B = \frac{1}{e^{-\beta\,\omega}-1}, \quad f_F = \frac{e^{-\beta\,\omega}}{1+e^{-\beta\,\omega}}. \tag{24}$$

Interpretation of the double Hilbert space.

The one particle state is build up from the thermal equilibrium state $o(\beta)\rangle$ by adding one particle without tilde or by eliminating one particle with tilde.

The particle with tilde is a hole of the physical particle (similar to the Dirac sea).



Normal equilibrium density matrix

- Assume a diagonal Hamiltonian:

$$H = \sum_{\{n_i\}=0}^{\infty} \left(\sum_{i=1}^{N} E_i\, n_i + E_0 \right) |n_1, \ldots, n_N\rangle \langle n_1, \ldots, n_N| \,, \qquad (25)$$

where $n_i = a_i^\dagger\, a_i$, and $\{n_i\} = \{n_i\}_{i=1}^{N} = n_1, \ldots, n_N$.

- Compute the relevant statistical quantities:

$$Z = Tr_{\{i\}} \left(e^{-\beta\, H} \right) = \sum_{\{\ell_i\}=0}^{\infty} \langle \{\ell_i\} | e^{-\beta\, H} | \{\ell_i\} \rangle = \prod_{i=1}^{N} \frac{e^{-\beta\, E_0}}{1 - e^{-\beta\, E_i}} = \prod_{i=1}^{N} \frac{e^{-K_0}}{1 - e^{-K_i}} \,. \qquad (26)$$

- The ordinary density matrix in equilibrium:

$$\rho_{eq} = \frac{e^{-\beta\, H}}{Z} = \frac{1}{Z} \sum_{\{n_i\}=0}^{\infty} e^{-\sum_{i=1}^{N} K_i\, n_i - K_0} \left| \{n_i\} \right\rangle \left\langle \{n_i\} \right| . \qquad (27)$$

Extended equilibrium density matrix

- Define a TFD statistical state, $|\Psi\rangle$, by

$$|\Psi\rangle = \sum_{\{n_i\}=0}^{\infty} \sqrt{\rho_{eq}}\; |\{n_i\}\rangle\; |\{\tilde{n}_i\}\rangle = \frac{1}{\sqrt{Z}} \sum_{\{n_i\}=0}^{\infty} e^{-\frac{1}{2}\left(\sum_{i=1}^{N} K_i\, n_i + K_0\right)} |\{n_i\}\rangle\; |\{\tilde{n}_i\}\rangle\,. \tag{28}$$

- The general representation theorem [M. Suzuki, J. Phys. Soc. Japan 54 no. 12, (1985)]:

The statistical state $|\Psi\rangle$ is independent of the chosen representation.

- The extended density operator is given by

$$\hat{\rho} = |\Psi\rangle\,\langle\Psi| = \frac{1}{Z} \sum_{\{n_i\}=0}^{\infty} \sum_{\{m_i\}=0}^{\infty} e^{-\frac{1}{2}\left(\sum_{i=1}^{N} K_i\,(n_i+m_i)+2\,K_0\right)} |\{n_i\}\rangle\,\langle\{m_i\}|\,|\{\tilde{n}_i\}\rangle\,\langle\{\tilde{m}_i\}|\,. \tag{29}$$

- Choose a bipartite system, namely

$$\{n_i\}_{i=1}^{N} = \{n_\mu\}_{\mu=1}^{p} \bigcup \{n_k\}_{k=p+1}^{N}\,, \quad p \leq N-1, \quad N \geq 2\,. \tag{30}$$

Partial density matrix and entanglement entropy

- Tracing over the parameters of the second system B:

$$\hat{\rho}_A = Tr_{\{B\}}\hat{\rho} = \sum_{\{\ell_k\}=0}^{\infty} \sum_{\{\tilde{\ell}_k\}=0}^{\infty} \langle\{\ell_k\}|\langle\{\tilde{\ell}_k\}|\hat{\rho}\,|\{\ell_k\}\rangle\,|\{\tilde{\ell}_k\}\rangle$$

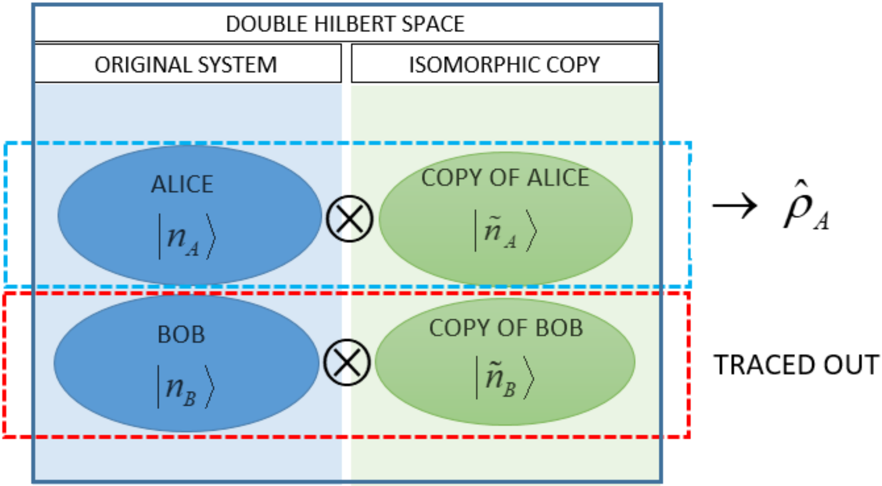
$$= \sum_{\{n_\mu\}=0}^{\infty} \sum_{\{m_\mu\}=0}^{\infty} e^{-\frac{1}{2}\sum_{\mu=1}^{p} K_\mu \left(2+n_\mu+m_\mu\right)} |\{n_\mu\}\rangle\,\langle\{m_\mu\}|\,|\{\tilde{n}_\mu\}\rangle\,\langle\{\tilde{m}_\mu\}|\,\prod_{\alpha=1}^{p}\left(e^{K_\alpha}-1\right). \quad (31)$$

- Finally, the extended renormalized entanglement entropy is given by

$$S_A\left(K_\mu\right) = -Tr_{\{A\}}\left(\hat{\rho}_A \ln \hat{\rho}_A\right)$$

$$= \frac{1}{2}\left(\prod_{\mu=1}^{p}\coth\frac{K_\mu}{4}\right)\sum_{\mu=1}^{p}\left\{K_\mu\left(1+\coth\frac{K_\mu}{4}\right)-2\ln\left(e^{K_\mu}-1\right)\right\}. \quad (32)$$

Schematic representation of the partial density matrix



Fisher information metric

- The parameter space: let $\vec{x}$ be a set of random variables from a real sample space X, then

a set of distributions $f(\vec{x},\vec{\theta})$, parametrized by $\vec{\theta}$, forms a statistical manifold.

- The Fisher information metric [J. Burbea, C. R. Rao, Probab.Math.Statist. 3 no. 2, (1984)]:

The Riemannian metric on this manifold is the Fisher information metric defined by the following Lebesgue integral:

$$g_{\mu\nu}(\vec{\theta}) = \int\limits_X \mathcal{D}_X\, f(\vec{x},\vec{\theta})\, \frac{\partial \ln f(\vec{x},\vec{\theta})}{\partial \theta^\mu}\, \frac{\partial \ln f(\vec{x},\vec{\theta})}{\partial \theta^\nu}\,. \tag{33}$$

- The only Riemannian metric is Fisher metric for which the geometry is invariant under coordinate transformations of $\vec{\theta}$ and also under one-to-one transformations of the random variable $\vec{x}$, [S. Amari, H. Nagaoka, AMS, 2007].

Fisher information metric

Following [H. Matsueda, arXiv:1408.5589 [hep-th]], one can define Fisher metric on the space parametrized by the inverse scaled temperatures K_μ by:

$$g_{\mu\nu} = \frac{\partial^2 S_A}{\partial K^\mu \, \partial K^\nu} = -\frac{F}{8} \left(A_\mu \, B_\nu + A_\nu \, B_\mu + C_{\mu\nu} + E \, D_{\mu\nu} \right) , \qquad (34)$$

$$A_\mu = 2 \operatorname{csch} \frac{K_\mu}{2} , \quad F = \prod_{\sigma=1}^{p} \coth \frac{K_\sigma}{4} , \qquad (35)$$

$$B_\mu = 1 + \coth \frac{K_\mu}{4} - \frac{K_\mu}{4} \operatorname{csch}^2 \frac{K_\mu}{4} - \frac{2}{1 - e^{-K_\mu}} , \qquad (36)$$

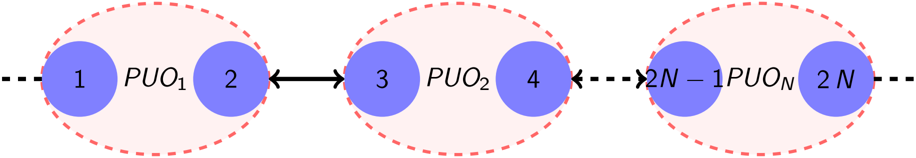
$$C_{\mu\nu} = \delta_{\mu\nu} \left[\left(2 - \frac{K_\mu}{2} \coth \frac{K_\mu}{4} \right) \operatorname{csch}^2 \frac{K_\mu}{4} + \frac{4}{1 - \cosh K_\mu} \right] , \qquad (37)$$

$$D_{\mu\nu} = 2 \operatorname{csch}^2 \frac{K_\nu}{4} \left(\delta_{\mu\nu} + \tanh \frac{K_\nu}{4} \sum_{\tau \neq \nu} \left\{ \delta_{\mu\tau} \operatorname{csch} \frac{K_\tau}{2} \right\} \right) , \qquad (38)$$

$$E = -\frac{1}{4} \sum_{\alpha=1}^{p} \left[K_\alpha \left(1 + \coth \frac{K_\alpha}{4} \right) - 2 \ln \left(e^{K_\alpha} - 1 \right) \right] . \qquad (39)$$

N minimally coupled $1d$ fourth-order PU oscillators

The diagonalization and quantization procedures can be found in [Dimov, Mladenov, Rashkov, Vetsov, Nuc. Phys. B 918 (2017), 317–336],



- The system 4-th order PUOs is described by the following Hamiltonian:

$$H_N = \frac{1}{2}\sum_{\mu=1}^{N}\sum_{k=0}^{1}\mathrm{sgn}(\alpha_{\mu,k})\left(p_\mu^k p_\mu^k + \omega_{\mu,k}^2 x_\mu^k x_\mu^k\right) + \frac{1}{2}\sum_{\langle\mu,\nu\rangle=1}^{N} c_{\mu\nu}x_\mu x_\nu. \quad (40)$$

- The Hamiltonian after diagonalization and quantization:

$$H_N = \sum_{j=1}^{2N}\hbar\lambda_j\left(\mathfrak{a}_j^\dagger\mathfrak{a}_j + \frac{1}{2}\right). \quad (41)$$

Fisher metric for two fourth-order PU oscillators

- The components of the $2d$ Fisher information metric for two minimally coupled fourth-order PU oscillators:

$$g_{11} = \frac{1}{64} \coth\frac{K_2}{4} \operatorname{csch}^2\frac{K_1}{4} \left[K_1 \left(3 + 5\coth^2\frac{K_1}{4} + 7\operatorname{csch}^2\frac{K_1}{4} \right) + 4\tanh\frac{K_1}{4} \right.$$
$$\left. + 4\coth\frac{K_1}{4} \left(K_1 + K_2 - 5 + K_2\coth\frac{K_2}{4} - 2\log\left[\left(e^{K_1} - 1\right)\left(e^{K_2} - 1\right) \right] \right) \right], \quad (42)$$

$$g_{12} = g_{21} = \frac{1}{32} \operatorname{csch}^2\frac{K_1}{4} \operatorname{csch}^2\frac{K_2}{4} \left[K_1 \left(1 + 2\coth\frac{K_1}{4} \right) + K_2 \left(1 + 2\coth\frac{K_2}{4} \right) - 4 \right.$$
$$\left. - 2\log\left[\left(e^{K_1} - 1\right)\left(e^{K_2} - 1\right) \right] \right], \quad (43)$$

$$g_{22} = \frac{1}{64} \coth\frac{K_1}{4} \operatorname{csch}^2\frac{K_2}{4} \left[K_2 \left(3 + 5\coth^2\frac{K_2}{4} + 7\operatorname{csch}^2\frac{K_2}{4} \right) + 4\tanh\frac{K_2}{4} \right.$$
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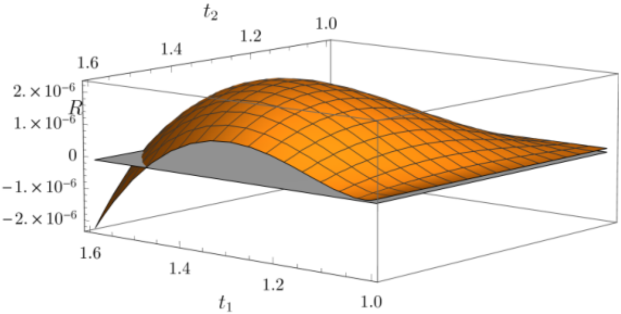
where $K_i = \beta\,\hbar\,\lambda_i$, $i = 1, 2$, are the inverse scaled temperatures.

Scalar curvature and second order phase transitions

- The elliptic case $R > 0$ $(t_i = e^{K_i})$:

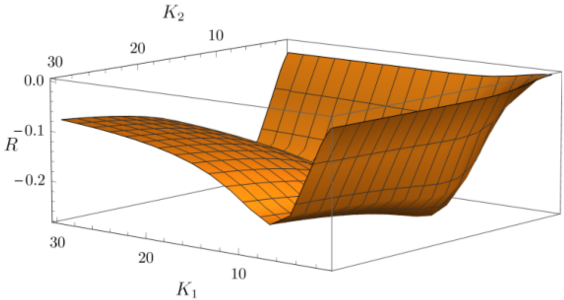


The local maximum of the scalar curvature corresponds to the maximum strength of the interaction between the components of the quasi-system.

The scalar curvature is free of divergencies, thus the quasi-system doesn't admit second order phase transitions.

Scalar curvature and second order phase transitions

- The hyperbolic case $R < 0$:



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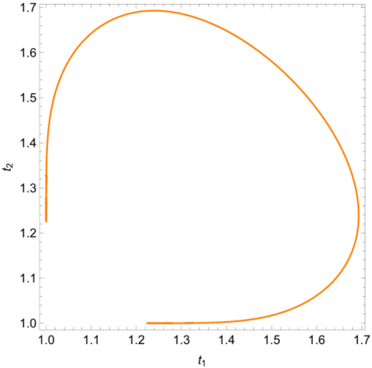
- The Ricci flat case $R = 0$:



The Ricci flat case $R = 0$ corresponds to a free non-interacting quasi-system.

Concluding remarks

- Begin with some Hamiltonian system.
- Apply diagonalization procedure $\rightarrow$ quasi-system (different Hamiltonian, same eigensystem).
- Apply quantization procedure $\rightarrow$ quasi-quantum system.
- TFD $\rightarrow$ double Hilbert space $\rightarrow$ extended entanglement entropy $\rightarrow$ Fisher information metric $\rightarrow$ metric invariants $\rightarrow$ phase structure.

Thank you!

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