Star product, star exponential and applications

Akira Yoshioka
Dept. Math. Tokyo University of Science, Japan

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Abstract

- We introduce star products for certain function space containing polynomials, and then we obtain an associative, non-commutative or commutative, algebras of functions.
- In this algebra we can consider exponential elements, which are called star exponentials.
- Using star exponentials we can define star functions in the star product algebra.
- This talk is not general, we explain just using concrete examples.

Background

- Weyl, Wigner, Moyal
- BFFLS Formal star product,
- Existence of formal star products
- non-formal star products

§1. Introduction: the idea

The idea of star product is deeply related to the canonical commutation relation in Quantum mechanics, which is given by a pair of operators $\hat{p}, \hat{q}$ such that

$$[\hat{p}, \hat{q}] = \hat{p}\,\hat{q} - \hat{q}\,\hat{p} = \sqrt{-1}\,\hbar = \mathrm{i}\,\hbar$$

where $\hat{p} = \mathrm{i}\,\hbar\,\partial_q$ and $\hat{q}$ is a multiplication operator $q\times$ acting on the functions of q, and $\hbar$ is the Planck constant.
The associative non-commutative algebra generated by $\hat{p}$ and $\hat{q}$ is called the *Weyl algebra* which plays a fundamental role in quantum mechanics.

We have another way to give the same algebra without using operators.

The idea is to introduce an associative product into the space of functions of (q, p).

The product is different from the usual multiplication of functions, but is given by a deformation of the usual multiplication in the following way.

(Cf. [1] Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer, *Deformation Theory and Quantization I, II*, Ann. Phys. **111** (1978) 61–151.
[8] Moyal, J. E.; Bartlett, M. S. *"Quantum mechanics as a statistical theory"*. Mathematical Proceedings of the Cambridge Philosophical Society. **45**(1949)).

The Poisson bracket and biderivation

For smooth functions f, g on $\mathbb{R}^2$ (or $\mathbb{C}^2$), we have the canonical Poisson bracket

$$\{f, g\}(q, p) = \partial_p f \partial_q g - \partial_q f \partial_p g, \quad (q, p) \in \mathbb{R}^2 \,(\text{or}\, \mathbb{C}^2)$$

In deformation quantization, we very often use the notation $\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}$ such as

$$\{f, g\} = f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) g = \partial_p f \ \partial_q g - \partial_q f \ \partial_p g$$

The Moyal product

The typical star product is the Moyal product given as follows.

For smooth functions f, g we consider a product $f *_o g$ given by a power series of the biderivation $\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p}$ such that

$$f *_O g = f \exp \frac{i\hbar}{2} \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) g = f \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right)^k g$$

$$= fg + \frac{i\hbar}{2} f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) g + \frac{1}{2!} \left(\frac{i\hbar}{2} \right)^2 f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right)^2 g$$

$$+ \cdots + \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k f \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right)^k g + \cdots$$

The product is well-defined when f or g is a polynomial, and it is easy to see that the product is associative for polynomials.

Canonical commutation relation

Now we calculate the commutator of the variables p and q. We see

$$p *_o q = p \exp \tfrac{i\hbar}{2} \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) q = p \sum_{k=0}^{\infty} \frac{1}{k!} \left(\tfrac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right)^k q$$

$$= pq + \tfrac{i\hbar}{2} p \left(\overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q} - \overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} \right) q = pq + \tfrac{i\hbar}{2}$$

Similarly we see

$$q *_o p = pq - \tfrac{i\hbar}{2}$$

Then p and q satisfy the canonical commutation relation under the commutator of the product $*_o$

$$[p, q]_* = p *_o q - q *_o p = i\hbar$$

The product $*_o$ is associative on polynomials with canonical commutation relation, and then we obtain the Weyl algebra given by the ordinary polynomials with the product $*_o$.

Using this Weyl algebra of the product $*_o$, we can obtain same results of quantum mechanics and some other extension.

In this talk, we give a brief review on the subject mainly related our investigation: Hideki Omori, Yoshiaki Maeda, Naoya Miyazaki, Akira Yoshioka:

[9] *Star exponential functions as two-valued elements*, *The breadth of symplectic and Poisson geometry*, Progress in Math. **232** (2005) 483–492.

[10] *Deformation of expressions for elements of an algebra*, *Symplectic, Poisson, and noncommutative geometry*, MSRI publications **62** (2014) 171–209.

Also see papers in arXiv: math-ph. 1307.0267, etc.

As an application of the star product algebra, we calculate the eigenvalues of the harmonic oscillator by means of the star product $*_O$.

Eigenvalues

The Schrödingier operator of the harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2}\left(\frac{\partial}{\partial q}\right)^2 + \frac{1}{2}q^2.$$

The eigenvalues are

$$E_n = \hbar(n + \tfrac{1}{2}),\ \ n = 0, 1, 2, \cdots$$

Star product calculation

We calculate these values E_n by means of the star product $*_o$ and functions of q and p, parallel to the method in quantum mechanics.

The classical hamiltonian function is

$$H = \frac{1}{2}(p^2 + q^2).$$

We put functions such as

$$a = \frac{1}{\sqrt{2\hbar}}(p + \mathrm{i}q), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}}(p - \mathrm{i}q).$$

Then we calculate the product explicitly and obtain

$$a^\dagger *_o a = \frac{1}{2\hbar}\left(p *_o p + \mathrm{i}[p,q]_* + q *_o q\right) = \frac{1}{2\hbar}\left(p \cdot p + \mathrm{i}\cdot\mathrm{i}\hbar + q \cdot q\right)$$

which shows $a^\dagger *_o a = \frac{1}{2\hbar}(p^2 + q^2) - \frac{1}{2}$ and then we have

$$H = \hbar(N + \tfrac{1}{2}), \quad (N = a^\dagger *_o a)$$

The commutator with respect to the star product is easily seen

$$[a, a^\dagger]_* = a *_o \ a^\dagger - a^\dagger *_o \ a = \frac{1}{2\hbar} 2(-\mathrm{i})[p, q]_* = 1$$

Now we set a function

$$f_0 = \frac{1}{\pi\hbar} \exp(-2\, aa^\dagger) = \frac{1}{\pi\hbar} \exp(-\tfrac{1}{\hbar}(p^2 + q^2))$$

and set a function

$$f_n = \frac{1}{n!} \underbrace{a^\dagger *_o \cdots *_o a^\dagger}_{n} *_o f_0 \ *_o \underbrace{a *_o \ \cdots *_o a}_{n}$$

for $n = 0, 1, 2, \cdots$. By a direct calculation we see

$$a *_o \ f_0 = f_0 *_o \ a^\dagger = 0$$

The relation $[a, a^\dagger]_* = a *_o a^\dagger - a^\dagger *_o a = 1$ induces $a *_o a^\dagger = a^\dagger *_o a + 1 = N + 1$ and a basic commutation relation

$$N *_o a^\dagger = (a^\dagger *_o a) *_o a^\dagger = a^\dagger *_o (a *_o a^\dagger) = a^\dagger *_o (N + 1)$$

Remark also that $a *_o f_0 = 0$ yields $N *_o f_0 = (a^\dagger *_o a) *_o f_0 = 0$. Then we calculate as

$$N *_o f_1 = N *_o (a^\dagger *_o f_0 *_o a) = a^\dagger *_o (N + 1) *_o f_0 *_o a = f_1$$

By a similar manner we easily see $N *_o f_k = f_k *_o N = k f_k$

Since $H = \hbar(N + \frac{1}{2})$ we have the solutions of the star eigenvalue problem

$$H *_o f_n = f_n *_o H = \hbar(n + \tfrac{1}{2}) f_n = E_n f_n, \quad (n = 0, 1, 2, \cdots)$$

and thus we obtain the eigenvalues of the harmonic oscillator $\hat{H}$.

§2.2. MIC-Kepler problem

Similarly the star product algebra also gives the exact eigenvalues and their multiplicities for the quantized Kepler problem and more general systems such as the MIC-Kepler problem, the Kepler problem under the influence of the Dirac magnetic monopole.

[4] Kanazawa T. and Yoshioka A., *Star Product and Its Application to the MIC-Kepler Problem*, J. Geom. Symmetry Phys. **25** (2012) 57–75.

Generalizing the derivation in the Moyal product, we give general star products as follows (cf. (Omori-Maeda-Miyazaki-Y [9])

§3.1. Examples: Moyal, normal, anti-normal products

The Moyal product is a well-known example of star product.

As in the previous section, we define: for polynomials f, g of the variables $(u_1, \dots, u_m, v_1, \dots, v_m)$, the Moyal product $f *_o g$ is given by the power series of the biderivation $\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) = \sum_j \left(\overleftarrow{\partial_{v_j}} \cdot \overrightarrow{\partial_{u_j}} - \overleftarrow{\partial_{u_j}} \cdot \overrightarrow{\partial_{v_j}}\right)$ such that

$$
\begin{aligned}
f *_O g &= f \exp \tfrac{i\hbar}{2}\left(\overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u} - \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}\right) g = f \sum_{k=0}^{\infty} \frac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u} - \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}\right)^k g \\
&= fg + \tfrac{i\hbar}{2} f \left(\overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u} - \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}\right) g + \tfrac{1}{2!}\left(\tfrac{i\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u} - \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}\right)^2 g \\
&\qquad + \cdots + \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k f \left(\overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u} - \overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}\right)^k g + \cdots
\end{aligned}
$$

Then we have

Theorem

The Moyal product is well-defined on polynomials, and associative.

Other typical star products are normal product $*_{_N}$, anti-nomal product $*_{_A}$ given similarly by

$$f *_{_N} g = f \exp \mathrm{i}\hbar \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right) g, \quad f *_{_A} g = f \exp -\mathrm{i}\hbar \left(\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g$$

These are also well-defined on polynomials and associative. By direct calculation we see easily the following.

Proposition

(i) *For these star products, the generators* $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ *satisfy the canoical commutation relations*

$$[u_k, v_l]_{*_L} = -\mathrm{i}\,\hbar\,\delta_{kl},\ [u_k, u_l]_{*_L} = [v_k, v_l]_{*_L} = 0,\ \ (k, l = 1, 2, \ldots, m)$$

where $*_{_L}$ *stands for* $*_{_O}, *_{_N}, *_{_A}$.

(ii) *Then the algebras* $(\mathbb{C}[u, v], *_L)$ $(L = O, N, A)$ *are mutually isomorphic and isomorphic to the Weyl algebra.*

Actually the algebra isomorphism

$$I_O^N : (\mathbb{C}[u,v], *_O) \to (\mathbb{C}[u,v], *_N)$$

is given explicitly by the power series of the differential operator such as

$$I_N^O\,(f) = \exp\left(-\tfrac{i\hbar}{2}\partial_u\partial_v\right)\,(f) = \sum_{l=0}^{\infty} \tfrac{1}{l!}(\tfrac{i\hbar}{2})^l\,(\partial_u\partial_v)^l\,\,(f)$$

And other isomorphisms are given in the similar form.

Remark

We remark here that these facts correspond to the well-known ordering problem in physics.

Now by generalizing the biderivations in the previous products, we define a star product on complex domain.

Biderivation

Let Λ be an arbitrary $n \times n$ complex matrix. We consider a biderivation on $\mathbb{C}^n$

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \cdots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \cdots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$$

where $(w_1, \cdots, w_n)$ is the coordinates of $\mathbb{C}^n$.

Now we define a star product by the power seires of the above biderivation such that

Definition

$$f *_\Lambda g = f \exp \frac{\mathrm{i}\hbar}{2} \left(\overleftarrow{\partial_w} \Lambda \overrightarrow{\partial_w} \right) g$$

Then similary as before we see easily

Theorem

For an arbitrary Λ, the star product $*_\Lambda$ is a well-defined associative product on complex polynomials.

Remark

(i) *The star product $*_\Lambda$ is a generalization of the previous products.*
Actually

- *if we put* $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *then we have the Moyal product*
- *if* $\Lambda = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$*, we have the normal product*
- *if* $\Lambda = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ *then the anti-normal product*

(ii) *If Λ is a symmetric matrix, the star product $*_\Lambda$ is commutative. Furthermore, if Λ is a zero matirx, then the star product is nothing but a usual commutative product.*

§3.3. Star product representation of the Weyl algebra

In this section, we fix the antisymmetric part of Λ in order to represent the Weyl algebra.

We assume the dimension is even, $n = 2m$. Let K be an arbitrary $2m \times 2m$ complex symmetric matrix. We put a completx matrix

$$\Lambda = J + K$$

where J is a fixed matrix such that

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since Λ is determined by the complex symmetric matrix K, we denote the star product by $*_K$ instead of $*_\Lambda$.

We consider polynomials of variables $(w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$. By an easy calculation one obtains for an arbitrary K

Proposition

(i) *For a star product $*_{_K}$, the generators $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ satisfy the canoical commutation relations*

$$[u_k, v_l]_* = -\mathrm{i}\,\hbar\delta_{kl},\ [u_k, u_l]_* = [v_k, v_l]_* = 0, \quad (k, l = 1, 2, \ldots, m)$$

(ii) *Then the algebra $(\mathbb{C}[u, v], *_{_K})$ is isomorphic to the Weyl algebra, and the algebra is regarded as a polynomial representation of the Weyl algebra.*

Equivalence

As in the case of typical star products, we have algebra isomorphisms as follows.

Proposition

For arbitrary star product algebras $(\mathbb{C}[u,v], *_{K_1})$ *and* $(\mathbb{C}[u,v], *_{K_2})$ *we have an algebra isomorphism* $I_{K_1}^{K_2} : (\mathbb{C}[u,v], *_{K_1}) \to (\mathbb{C}[u,v], *_{K_2})$ *given by the power series of the differential operator* $\partial_w(K_2 - K_1)\partial_w$ *such that*

$$I_{K_1}^{K_2}(f) = \exp\left(\frac{i\hbar}{4}\partial_w(K_2 - K_1)\partial_w\right)(f)$$

where $\partial_w(K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}$.

By a direct calculation we have

Theorem

Then isomorphisms satisfy the following chain rule:

1. $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = Id$
2. $\left(I_{K_1}^{K_2}\right)^{-1} = I_{K_2}^{K_1}$

Remark

1. *By the previous proposition we see the algebras* $(\mathbb{C}[u,v], *_{K})$ *are mutually isomorphic and isomorphic to the Weyl algebra. Hence we have a family of star product algebras* $\{(\mathbb{C}[u,v], *_{K})\}_K$ *where each element is regarded as a polynomial representation of the Weyl algebra.*
2. *The above equivalences also exist between star products* $*_{\Lambda}$ *for arbitrary* Λ*'s with a common skew symmetric part.*

§3.4. Star exponentials

Using polynomial expressions, we can consider exponential elements in the star product algebra.

Idea of definition. Here we are considering general star product $*_\Lambda$. For a polynomial H_* of the star product algebra, we want to define a star exponential $e_*^{t\frac{H_*}{i\hbar}}$. However, the expansion $\sum_n \frac{t^n}{n!}\left(\frac{H_*}{i\hbar}\right)^n$ of power series of $\frac{H_*}{i\hbar}$ with respect to the star product $*_\Lambda$ is not convergent in general.

Then we define a star exponential by the differential equation.

Definition

The star exponential $e_*^{t\frac{H_*}{i\hbar}}$ *is given as a solution of the differential equation*

$$\frac{d}{dt}F_t = \frac{H_*}{i\hbar} *_\Lambda F_t, \quad F_0 = 1.$$

Examples

We are interested in the star exponentials of linear polynomials, and quadratic ones. For simplicity we consider the case $\Lambda = J + K$. For these polynomials, for example we have the following explicit solutions for $*_{\Lambda} = *_{K}$.

Lienar case

We denote a linear polynomial by $l = \sum_{j=1}^{2m} a_j w_j$. We see

Proposition

For $l = \sum_j a_j w_j =< \boldsymbol{a}, \boldsymbol{w} >$, the star exponential with respect to the product $*_{\Lambda}$ is

$$e_{*_{\Lambda}}^{t(l/i\hbar)} = e^{t^2 \boldsymbol{a} K \boldsymbol{a}/4i\hbar} e^{t(l/i\hbar)}$$

Quadratic case

Proposition

For a quadratic polynomial $Q_* = \langle \boldsymbol{w}A, \boldsymbol{w}\rangle_*$ *where* A *is a* $2m \times 2m$ *complex symmetric matrix, we have*

$$e_{*_\Lambda}^{t(Q_*/i\hbar)} = \frac{2^m}{\sqrt{\det(I - \kappa + e^{-2t\alpha}(I + \kappa))}} e^{\frac{1}{i\hbar}\langle \boldsymbol{w} \frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J, \boldsymbol{w}\rangle}$$

where $\kappa = KJ$ *and* $\alpha = AJ$*.*

§3.5. Star functions

By the same way as in the ordinary exponential functions, we can obtain several non-commutative or commutative functions using star exponentials.

There are many application of star exponential functions. Today we show examples using a linear star exponentials. (More details, see for example; Omori Maeda Miyazaki Yoshioka Deformation of expression of elements of algebras, MSRI publication **62** (2014). Also see ArXiv OMMY math-ph. e.g., 1307.0267)

In what follows, we consider the star product for the simplest case where $n \times n$ matrix is of the form

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0_{n-1} \end{pmatrix}$$

Then we see easily that the star product is commutative and explicitly given by $f *_{\Lambda} g = f \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{w_1}}\overrightarrow{\partial_{w_1}}\right) g$.

This means that the algebra is essentially reduced to space of functions of one varible w_1.
Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2} \overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w)$$

A direct calculation gives that the star exponential of $\mathrm{i}\, tw$ with respect to $*_\tau$ is

Proposition

$$\exp_{*_\tau} \mathrm{i}\, tw = \exp(\mathrm{i}\, tw - (\tau/4) t^2)$$

§3.5.1. Star Hermite function

Recall the identity

$$\exp\left(\sqrt{2}tw - \tfrac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w)\frac{t^n}{n!}$$

where $H_n(w)$ is an Hermite polynomial. By the explicit formula $\exp_{*_\tau} \mathrm{i}\, tw = \exp(\mathrm{i}\, tw - (\tau/4)t^2)$, we see

$$\exp_*(\sqrt{2}tw_*)_{\tau=-1} = \exp\left(\sqrt{2}tw - \tfrac{1}{2}t^2\right)$$

Since $\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty}(\sqrt{2}w_*)^n \frac{t^n}{n!}$ we have

$$H_n(w) = (\sqrt{2}w_*)^n_{\tau=-1}$$

Star Hermite function We define $*$-Hermite function by

$$H_n(w,\tau) = (\sqrt{2}w_*)^n, \quad (n = 0, 1, 2, \cdots)$$

with respect to $*_\tau$ product. Then we have

$$\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty} H_n(w,\tau)\,\frac{t^n}{n!}$$

Identities

Trivial identity $\frac{d}{dt}\exp_*(\sqrt{2}tw_*) = \sqrt{2}w * \exp_*(\sqrt{2}tw_*)$ for the product $*_\tau$ yields the identity

$$\frac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}wH_n(w,\tau) = H_{n+1}(w,\tau), \quad (n = 0, 1, 2, \cdots).$$

for every $\tau \in \mathbb{C}$.

The exponential law

$$\exp_{*_\tau}(\sqrt{2}sw_*) * \exp_{*_\tau}(\sqrt{2}tw_*) = \exp_{*_\tau}(\sqrt{2}(s+t)w_*)$$

for the product $*_\tau$ yields the identity

$$\sum_{k+l=n} \tfrac{n!}{k!l!} H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau).$$

for every $\tau \in \mathbb{C}$.

We can express the Jacobi's theta functions by using star exponentials.

Recall the formula

$$\exp_{*_\tau} i\,tw = \exp(i\,tw - (\tau/4)t^2)$$

Hence for $\mathrm{Re}\,\tau > 0$, the star exponential $\exp_{*_\tau} ni\,w = \exp(ni\,w - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then the summation converges to give

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2ni\,w = \sum_{n=-\infty}^{\infty} \exp\left(2ni\,w - \tau\,n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni\,w}, \quad (q = e^{-\tau})$$

This is convergent and gives Jacobi's theta function $\theta_3(w,\tau)$. Similarly we have expressions of theta functions as

$$\theta_{1*_\tau}(w) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau}(2n{+}1)i\,w, \quad \theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau}(2n{+}1)i\,w$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau} 2ni\,w, \quad \theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau} 2ni\,w$$

Remark that $\theta_{k*_\tau}(w)$ is the Jacobi's theta function $\theta_k(w,\tau)$, $k = 1,2,3,4$ respectively.

We have trivial identities because of the exponential law

$$\exp_{*_\tau} 2i\, w *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$\exp_{*_\tau} 2i\, w *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

Then using $\exp_{*_\tau} 2i\, w = e^{-\tau} e^{2i\, w}$ and the product formula directly we see the above identities are just

$$e^{2i\, w-\tau}\theta_{k*_\tau}(w + i\,\tau) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$e^{2i\, w-\tau}\theta_{k*_\tau}(w + i\,\tau) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

§3.5.3. $*$-delta functions

Since the $*_\tau$-exponential $\exp_{*_\tau}(itw_*) = \exp(itw - \frac{\tau}{4}t^2)$ is raidly decreasing with respect to t when $\mathrm{Re}\,\tau > 0$. Then the integral

$$\int_{-\infty}^{\infty} \exp_{*_\tau}(it(w-a)_*)\,dt = \int_{-\infty}^{\infty} \exp(it(w-a) - \frac{\tau}{4}t^2)dt$$

converges for any $a \in \mathbb{C}$. We put a star δ-function

$$\delta_*(w-a) = \int_{-\infty}^{\infty} \exp_{*_\tau}(it(w-a)_*)dt$$

which has a meaning at τ with $\mathrm{Re}\,\tau > 0$. It is easy to see for any element $p_*(w) \in (\mathbb{C}[w], *_\tau)$,

$$p_*(w) *_\tau \delta_*(w-a) = p(a)\delta_{*_\tau}(w-a),\ w_* *_\tau \delta_*(w) = 0.$$

Using the Fourier transform we have

Proposition

$$\theta_{1*}(w) = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + \tfrac{\pi}{2} + n\pi)$$

$$\theta_{2*}(w) = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + n\pi)$$

$$\theta_{3*}(w) = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \delta_*(w + n\pi)$$

$$\theta_{4*}(w) = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \delta_*(w + \tfrac{\pi}{2} + n\pi).$$

Now, we consider the τ with the condition $\mathrm{Re}\ \tau > 0$. Then we calcultate the integral and obtain $\delta_*(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w-a)^2\right)$ Then we have

$$\begin{aligned}\theta_3(w,\tau) &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w+n\pi) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w+n\pi)^2\right) \\ &= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \sum_{n=-\infty}^{\infty} \exp\left(-2n\frac{1}{\tau}w - \frac{1}{\tau}n^2\tau^2\right) \\ &= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \theta_{3*}(\frac{2\pi w}{i\tau}, \frac{\pi^2}{\tau}).\end{aligned}$$

We also have similar identities for other $*$-theta functions by the similar way.

- linear case : star special functions, star Eisenstein series.
- quadratic case: group like object, singularities. etc.

Appendix : MIC-Kepler probelm

Background

McIntosh and Cisneros [7] studied the dynamical system describing the motion of a charged particle under the influence of Dirac's monopole field besides the Coulomb's potential.
Iwai-Uwano [2] gives the Hamiltonian description for the MIC-Kepler problem.

Point

Iwai-Uwano showed that the classical system of MIC-Kepler problem is obtained by the geometric method of S^1-reduction, or Marsden-Weinstein reduction for symplectic manifolds.

Star product method uses only classical system with deformed product.
Then by the star product calculation, we expect to deal with the quantized system of MIC-Kepler problem by means of the geometric method of Marsden-Weinstein reduction in natural way.

We discuss this in this subsection.

MIC-Kepler problem

Now we introduce the MIC-Kepler problem.

We consider a closed two form on $\dot{\mathbb{R}}^3 = \mathbb{R}^3 - \{\mathbf{0}\}$ such that

$$\Omega = (\, q_1 \, \mathrm{d}q_2 \wedge \mathrm{d}q_3 \, + \, q_2 \, \mathrm{d}q_3 \wedge \mathrm{d}q_1 \, + \, q_3 \, \mathrm{d}q_1 \wedge \mathrm{d}q_2)/r^3$$

where $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$. We consider the cotangent bundle $T^*\dot{\mathbb{R}}^3$ and a symplectic form

$$\sigma_\mu = \mathrm{d}p_1 \wedge \mathrm{d}q_1 \, + \, \mathrm{d}p_2 \wedge \mathrm{d}q_2 \, + \, \mathrm{d}p_3 \wedge \mathrm{d}q_3 \, + \Omega_\mu$$

where $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, q_2, q_3, p_1, p_2, p_3) \in T^*\dot{\mathbb{R}}^3$ and the 2-form $\Omega_\mu \equiv \mu\,\Omega$ stands for Dirac's monopole field of strength $\mu \in \mathbb{R}$.

Then the MIC-Kepler problem is given as the triple

$$(T^*\dot{\mathbb{R}}^3, \sigma_\mu , H_\mu)$$

where H_μ is the Hamiltonian function such that

$$H_\mu (\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{\mu^2}{2r^2} - \frac{k}{r}$$

and k is a positive constant.

When $\mu = 0$ the system is just the Kepler problem.

S^1-action

The MIC-Kepler problem is obtained by the S^1-reduction from the conformal Kepler problem on $T^*\dot{\mathbb{R}}^4$ (Iwai-Uwano [2]) as follows.

We denote the points by $y \in \mathbb{R}^4$ and $(y, \eta) \in T^*\mathbb{R}^4$.

We identify the point of $T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4)$ by

$$T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4) \mapsto (z_1, z_2, \zeta_1, \zeta_2) \in T^*\mathbb{C}^2 = \mathbb{C}^4$$

where

$$z_1 = y_1 + \mathrm{i}\, y_2, \quad z_2 = y_3 + \mathrm{i}\, y_4, \quad \zeta_1 = \eta_1 + \mathrm{i}\, \eta_2, \quad \zeta_2 = \eta_3 + \mathrm{i}\, \eta_4$$

The canonical one form θ on $T^*\mathbb{R}^4$ is written as

$$\theta(z,\zeta) = \mathrm{Re}\,(\bar{\zeta}\cdot \mathrm{d}z).$$

The S^1 action on the cotangent bundle $T^*\dot{\mathbb{R}}^4$ is given by

$$\varphi_t : (z,\zeta) \mapsto (\mathrm{e}^{\mathrm{i}t}z, \mathrm{e}^{\mathrm{i}t}\zeta), \quad (t \in \mathbb{R})$$

which preserves the canonical one form θ and then is an exact symplectic action.

The induced vector field $v(z,\zeta)$ on $T^*\dot{\mathbb{R}}^4$ of the action is

$$v(z,\zeta) = (\mathrm{i}z, \mathrm{i}\,\zeta)$$

and then a moment map ψ of the action is given by

$$\psi(z,\zeta) = \iota_v\theta(z,\zeta) = \mathrm{Im}\ \zeta\cdot\bar{z} = (\zeta\cdot\bar{z} - \bar{\zeta}\cdot z)/2\,\mathrm{i}$$

S^1-reduction

Following the Marsden-Weinstein reduction theroy, we consider a level set of the moment map $\psi^{-1}(\mu)$ for $\mu \in \mathbb{R}$. Then the S^1-bundle $\pi_\mu : \psi^{-1}(\mu) \to \psi^{-1}(\mu)/S^1$ has the symplectic structure ω_μ such that $\iota_\mu^* \mathrm{d}\theta = \pi_\mu^* \omega_\mu$, hence we have a reduced symplectic manifold $(\psi^{-1}(\mu)/S^1, \omega_\mu)$, where $\iota_\mu : \psi^{-1}(\mu) \to T^*\dot{\mathbb{R}}^4$ is the inclusion map. Then one can show

Proposition (Iwai-Uwano [2])

The reduced phase space is diffeomorphic to the symplectic manifold of the MIC-Kepler problem,

$$(\psi^{-1}(\mu)/S^1, \omega_\mu) \simeq (T^*\dot{\mathbb{R}}^3, \sigma_\mu)$$

Conformal Kepler problem on $T^*\dot{\mathbb{R}}^4$

Now we consider a harmonic oscillator on $T^*\dot{\mathbb{R}}^4$

$$H(z,\zeta) = \frac{1}{2}\,|\zeta|^2 + \frac{1}{2}\,\omega^2\,|z|^2$$

Iwai-Uwano [2] introduces the conformal Kepler problem with the Hamiltonian

$$H_{CF}(z,\zeta) = \frac{1}{4|z|^2}\,(H(z,\zeta) - 4k) - \frac{1}{8}\omega^2 = \frac{1}{8|z|^2}|\zeta|^2 - \frac{k}{|z|^2}$$

The MIC-Kepler problem is the reduced hamitonian system of the conformal Kepler problem, i.e.,

$$\pi_\mu^* H_\mu = \iota_\mu^* H_{CF}$$

The conformal Kepler problem is related to the harmonic oscillator on $T^*\dot{\mathbb{R}}^4$ as

$$4|z|^2\left(H_{CF}(z,\zeta)+\tfrac{1}{8}\omega^2\right)=H(z,\zeta)-4k$$

Hence the energy surfaces in $T^*\dot{\mathbb{R}}^4$ coincide, i.e.,

$$H_{CF}=-\tfrac{1}{8}\omega^2 \Longleftrightarrow H=4k.$$

Star product calculation of the eigenvalues

On 8-dimensional phase space $T^*\dot{\mathbb{R}}^4$, we have the canonical Poisson bracket and then by the same way as the previous section, we have the star product $*_o$.

We consider functions

$$\begin{cases} b_1(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, z_1 + \frac{i}{\sqrt{\omega\hbar}}\, \zeta_1\right), \quad b_1(z,\zeta)^\dagger = \overline{b_1(z,\zeta)}, \\ b_2(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, z_2 + \frac{i}{\sqrt{\omega\hbar}}\, \zeta_2\right), \quad b_2(z,\zeta)^\dagger = \overline{b_2(z,\zeta)}, \\ b_3(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_1 + \frac{i}{\sqrt{\omega\hbar}}\, \bar{\zeta}_1\right), \quad b_3(z,\zeta)^\dagger = \overline{b_3(z,\zeta)}, \\ b_4(z,\zeta) &= \frac{1}{2}\left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_2 + \frac{i}{\sqrt{\omega\hbar}}\, \bar{\zeta}_2\right), \quad b_4(z,\zeta)^\dagger = \overline{b_4(z,\zeta)}. \end{cases}$$

We see the commutators of these functions are

$$[b_j, b_k]_* = [b_j^\dagger, b_k^\dagger]_* = 0, \quad [b_j, b_k^\dagger]_* = \delta_{jk} \quad (j, k = 1, 2, 3, 4).$$

We set

$$N = b_1^\dagger *_o b_1 + b_2^\dagger *_o b_2 + b_3^\dagger *_o b_3 + b_4^\dagger *_o b_4.$$

Then we see

$$H = \hbar\omega\,(N + 2),$$

and the moment map $\psi(z, \zeta)$ is written in terms of b_j, $b_j^\dagger$ as

$$\psi(z, \zeta) = \frac{\hbar}{2}\,(-b_1^\dagger *_o b_1 - b_2^\dagger *_o b_2 + b_3^\dagger *_o b_3 + b_4^\dagger *_o b_4).$$

We put for $j = 1, 2, 3, 4$

$$f_{j,0}(z,\zeta) = \frac{1}{\pi\hbar} e^{-2\, b_j^\dagger\, b_j}, \quad f_{j,k}(z,\zeta) = \frac{1}{k!}\, (b_j^\dagger)_*^k *_O f_{j,0} *_O (b_j)_*^k.$$

We consider

$$f_{\vec{n}} = f_{1,n_1} *_O f_{2,n_2} *_O f_{3,n_3} *_O f_{4,n_4}, \quad \vec{n} = (n_1, n_2, n_3, n_4).$$

Parallel to Iwai-Uwano [3], we can calculate the eigenvalues of the MIC-Kepler problem as follows.
Similarly as before we easily see

$$H *_O f_{\vec{n}} = \hbar\omega(N+2) *_O f_{\vec{n}} = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2)\, f_{\vec{n}}$$

and

$$\psi *_O f_{\vec{n}} = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)\, f_{\vec{n}}$$

Hence the energy level

$$H_{CF} = -\frac{1}{8}\omega^2 \Longleftrightarrow H = 4k \quad \text{and} \quad \psi = \mu$$

is quantized as

$$4k = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2)$$

and

$$\mu = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)$$

Thus the quantized energy level of H_{CF} is $-\frac{1}{8}\omega^2 = -\frac{2k^2}{\hbar^2(n_1+n_2+n_3+n_4+2)^2}$ and the strength of magnetic monopole is quantized as $\mu = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)$.

Thus we have

Theorem

The eigenvalues of the MIC-Kepler problem with the strength of magnetic monople $\hbar \frac{m}{2}$ is

$$E_n = -\frac{2k^2}{\hbar^2(n+2)^2}, \quad (\, n \geq |m|, \quad \textit{and} \quad n \pm m \equiv 0 \mod 2\,).$$

The multiplicity of the eigenvalue E_n is

$$\frac{(n+m+2)(n-m+2)}{4}$$

This is the same as the ones in Iwai-Uwano [3].

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