The Heisenberg group and $SL_2(\mathbb{R})$

a survival pack

Vladimir V. Kisil

School of Mathematics
University of Leeds (England)

email: kisilv@maths.leeds.ac.uk
Web: http://www.maths.leeds.ac.uk/~kisilv

Geometry, Integrability, Quantization–2018, Varna

Fractional Linear Transformations

and cycles

Symmetries of Lie spheres geometry include fractional linear transformations (FLT) of the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ x \mapsto \frac{ax+b}{cx+d}, \qquad \text{where } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0. \tag{1}$$

Cycles (quadrics) in $\mathbb{R}^{pq}$ given by FSC 2×2 matrices:[1]

$$k\bar{x}x - l\bar{x} - x\bar{l} + m = 0 \quad \leftrightarrow \quad C = \begin{pmatrix} l & m \\ k & \bar{l} \end{pmatrix}, \tag{2}$$

where $k, m \in \mathbb{R}$ and $l \in \mathbb{R}^{pq}$. For brevity we also encode a cycle by its coefficients (k, l, m). A justification of (2) is provided by the identity:

$$\begin{pmatrix} 1 & \bar{x} \end{pmatrix} \begin{pmatrix} l & m \\ k & \bar{l} \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = kx\bar{x} - l\bar{x} - x\bar{l} + m, \qquad \text{since } \bar{x} = -x \text{ for } x \in \mathbb{R}^{pq}.$$

[1] Fillmore and Springer, "Möbius groups over general fields using Clifford algebras associated with spheres", 1990; Cnops, *An introduction to Dirac operators on manifolds*, 2002, (4.12); Kisil, *Geometry of Möbius Transformations: Elliptic, Parabolic and Hyperbolic Actions of* $SL_2(\mathbf{R})$, 2012, § 4.4.

FLT invariant inner product

The identification is also FLT-covariant in the sense that the transformation $x \mapsto \frac{ax+b}{cx+d}$ (1) associated with the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends a cycle C to the cycle MCM^*.[2]

The FLT-invariant inner product of cycles C_1 and C_2 is

$$\langle C_1, C_2 \rangle = \Re \operatorname{tr}(C_1 C_2)\,, \tag{3}$$

where $\Re$ denotes the scalar part of a Clifford number. This definition in term of matrices immediately implies that the inner product is FLT-invariant. The explicit expression in terms of components of cycles $C_1 = (k_1, l_1, m_1)$ and $C_2 = (k_2, l_2, m_2)$ is also useful sometimes:

$$\langle C_1, C_2 \rangle = l_1 l_2 + \bar{l}_1 \bar{l}_2 + m_1 k_2 - m_2 k_1\,. \tag{4}$$

All non-linear conditions below can be linearised if the additional *quadratic condition of normalisation type is imposed*:

$$\langle C, C \rangle = \pm 1. \tag{5}$$

[2] Cnops, *An introduction to Dirac operators on manifolds*, 2002, (4.16).

Inner product and geometric relations I

The relation $\langle \mathsf{C}_1, \mathsf{C}_2 \rangle = 0$ is called the *orthogonality* of cycles. In most cases it corresponds to orthogonality of quadrics in the point space. It is a part of the following list:

1. A quadric is *flat* (i.e. is a hyperplane), that is, its equation is linear.
   1.1 k component of the cycle vector is zero;
   1.2 is orthogonal $\langle \mathsf{C}_1, \mathsf{C}_\infty \rangle = 0$ to the "zero-radius cycle at infinity" $\mathsf{C}_\infty = (0, 0, 1)$.
2. A quadric is a *Lobachevsky line* if it is orthogonal $\left\langle \mathsf{C}_1, \mathsf{C}_{\mathbb{R}} \right\rangle = 0$ to the real line cycle $\mathsf{C}_{\mathbb{R}}$. A similar condition is meaningful in higher dimensions as well.
3. A quadric C represents a *point*, that is, it has zero radius at given metric of the point space. Then, the determinant of the corresponding FSC matrix is zero or, equivalently, the cycle is self-orthogonal (isotropic): $\langle \mathsf{C}, \mathsf{C} \rangle = 0$.
4. Two quadrics are *orthogonal* in the point space $\mathbb{R}^{pq}$. Then, cycles are orthogonal in the sense of the inner product (3).

Inner product and geometric relations II

5. Two cycles C and $\tilde{\mathsf{C}}$ are *tangent*

$$\left\langle \mathsf{C}, \tilde{\mathsf{C}} \right\rangle^2 = \langle \mathsf{C}, \mathsf{C} \rangle \left\langle \tilde{\mathsf{C}}, \tilde{\mathsf{C}} \right\rangle . \tag{6}$$

   If cycle C is normalised by the condition (5), it is linear to components of the cycle C:

$$\left\langle \mathsf{C}, \tilde{\mathsf{C}} \right\rangle = \pm \sqrt{\left\langle \tilde{\mathsf{C}}, \tilde{\mathsf{C}} \right\rangle}. \tag{7}$$

   Different signs here represent internal and outer touch.

6. *Inversive distance* θ of two (non-isotropic) cycles is defined by the formula:

$$\left\langle \mathsf{C}, \tilde{\mathsf{C}} \right\rangle = \theta \sqrt{\langle \mathsf{C}, \mathsf{C} \rangle} \sqrt{\left\langle \tilde{\mathsf{C}}, \tilde{\mathsf{C}} \right\rangle} \tag{8}$$

   In particular, the above discussed orthogonality corresponds to $\theta = 0$ and the tangency to $\theta = \pm 1$. For intersecting spheres θ provides the cosine of the intersecting angle.

Inner product and geometric relations III

7. A generalisation of *Steiner power* d of two cycles is defined as:[3]

$$\mathsf{d} = \left\langle \mathsf{C}, \tilde{\mathsf{C}} \right\rangle + \sqrt{\langle \mathsf{C}, \mathsf{C} \rangle} \sqrt{\left\langle \tilde{\mathsf{C}}, \tilde{\mathsf{C}} \right\rangle}, \tag{9}$$

where both cycles C and $\tilde{\mathsf{C}}$ are k-normalised, that is the coefficient in front the quadratic term in (2) is 1. Geometrically, the generalised Steiner power for spheres provides the square of tangential distance. However, this relation is again non-linear for the cycle C.
If we replace C by the cycle $\mathsf{C}_1 = \frac{1}{\sqrt{\langle \mathsf{C}, \mathsf{C} \rangle}} \mathsf{C}$ satisfying (5), the identity (9) becomes:

$$\mathsf{d} \cdot \mathsf{k} = \left\langle \mathsf{C}_1, \tilde{\mathsf{C}} \right\rangle + \sqrt{\left\langle \tilde{\mathsf{C}}, \tilde{\mathsf{C}} \right\rangle}, \tag{10}$$

where $\mathsf{k} = \frac{1}{\sqrt{\langle \mathsf{C}, \mathsf{C} \rangle}}$ is the coefficient in front of the quadratic term of C_1. The last identity is linear in terms of the coefficients of C_1.

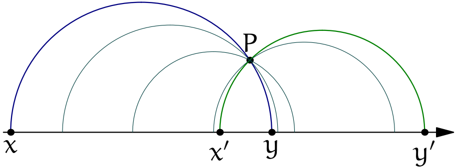
[3] Fillmore and Springer, "Determining Circles and Spheres Satisfying Conditions Which Generalize Tangency", 2000, § 1.1.

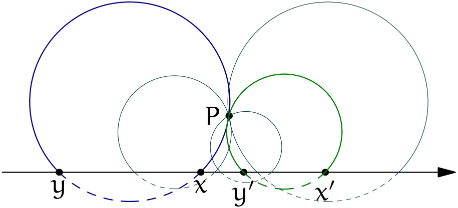
Ensembles of cycles: Poincaré extension

The *Poincaré extension* of Möbius transformations from the real line to the upper half-plane of complex numbers is described by a triple of cycles $\{C_1, C_2, C_3\}$ such that:[4]

1. C_1 and C_2 are orthogonal to the real line;
2. $\langle C_1, C_2\rangle^2 \leqslant \langle C_1, C_1\rangle \langle C_2, C_2\rangle$;
3. C_3 is orthogonal to any cycle in the triple including itself.

A modification with ensembles of four cycles describes an extension from the real line to the upper half-plane of complex, dual or double numbers. The construction can be generalised to arbitrary dimensions.





[4] Kisil, "Poincaré Extension of Möbius Transformations", 2017.

Poincaré extension: parabolic and hyperbolic

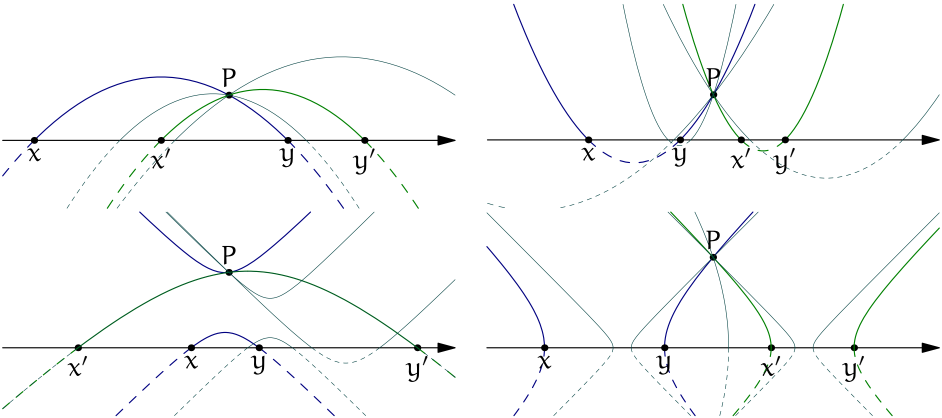


Figure: Poincaré extensions: first column presents points defined by the intersecting intervals $[x, y]$ and $[x', y']$, the second column—by disjoint intervals. Each row uses the same type of conic sections—circles, parabolas and hyperbolas respectively.

Logarithmic spirals: a universal pattern

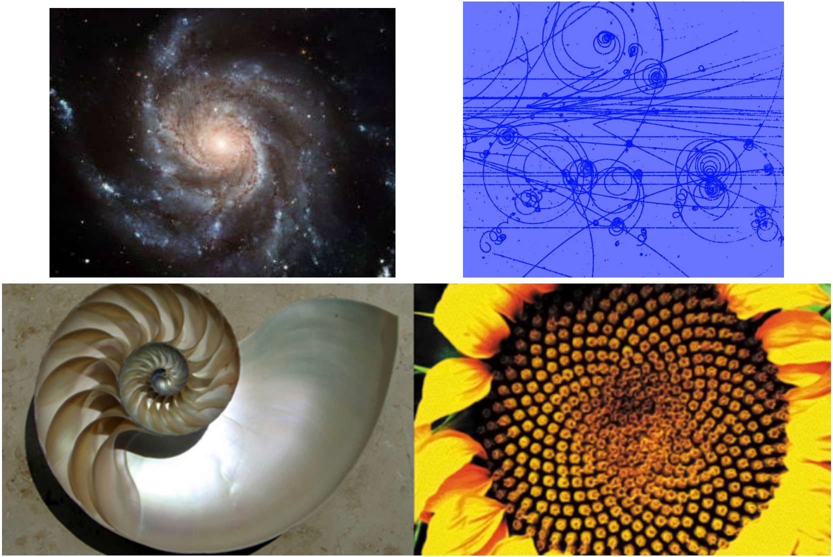


Figure: Natural logarithmic spirals: galaxies, traces of elementary particles, seashells and sunflowers.

Logarithmic spirals and loxodromes

Logarithmic spirals are integral curves of the fundamental differential equation $\dot{y} = \lambda y$, $\lambda \in \mathbb{C}$—a first approximation to many natural processes. Thus, images of logarithmic spirals under FLT, called *loxodromes* are not rare: from the stereographic projection of a *rhumb line* to archetypal Carleson arc.[5]

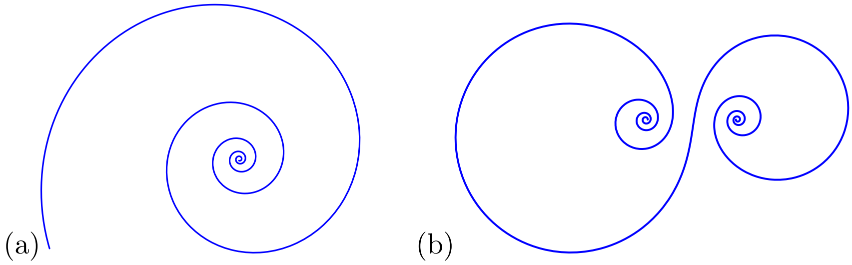


Figure: A logarithmic spiral (a) and its image under FLT—loxodrome (b).

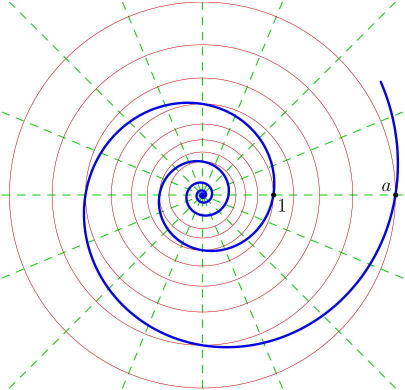
[5] Böttcher and Karlovich, "Cauchy's singular integral operator and its beautiful spectrum", 2001; Bishop et al., "Local Spectra and Index of Singular Integral Operators with Piecewise Continuous Coefficients on Composed Curves", 1999.

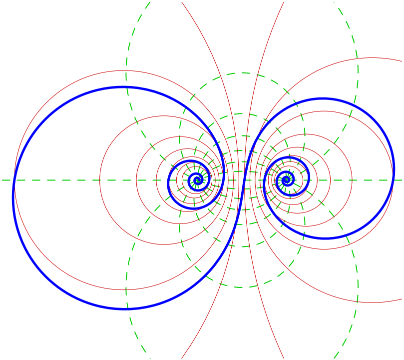
Ensembles of cycles: loxodromes

Loxodromes are parametrised by a triple of cycles $\{C_1, C_2, C_3\}$ s.t.:[6]

1. C_1 is orthogonal to C_2 and C_3;
2. $\langle C_2, C_3\rangle^2 \geqslant \langle C_2, C_2\rangle \langle C_3, C_3\rangle$.

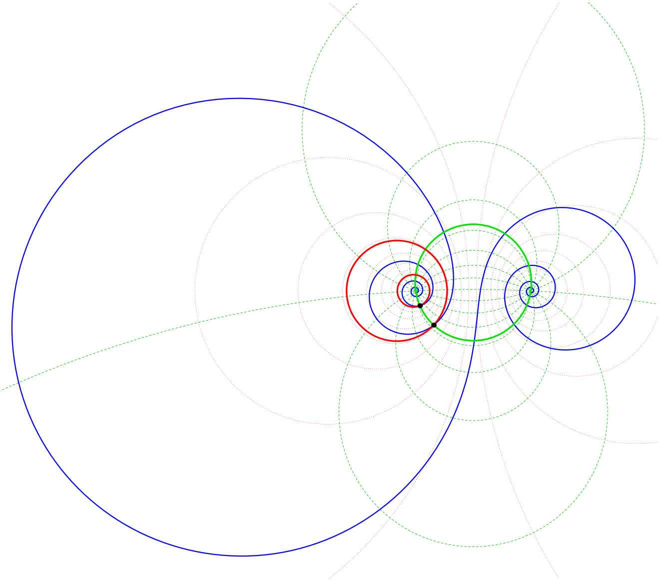
Then, main invariant properties of Möbius–Lie geometry, e.g. tangency of loxodromes, can be expressed in terms of this parametrisation.





[6] Kisil and Reid, "Conformal Parametrisation of Loxodromes by Triples of Circles", 2018.

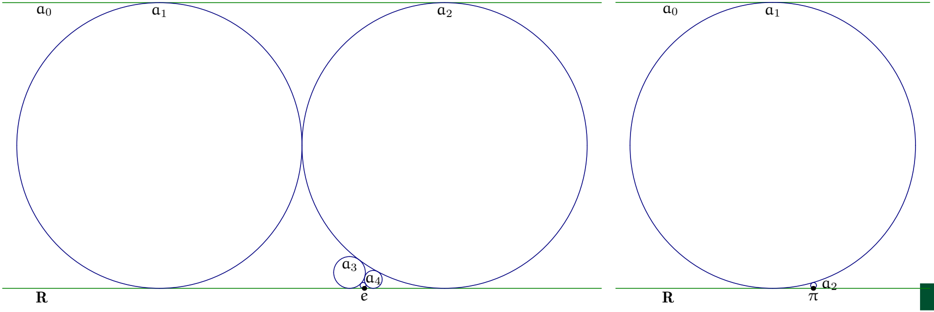
Animated parametrisation of loxodromes



Continued fractions

Continued fractions are iterations of FLT—chains of tangent horocycles:[7]

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}, \qquad \pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \dots}}}}.$$



[7] Beardon and Short, “A Geometric Representation of Continued Fractions”, 2014.

Continued fractions and FLT

Continued fractions are composition of specific FLT

$$S(n) = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = s_1 \circ s_2 \circ \ldots \circ s_n, \qquad \text{where} \quad s_j(z) = \frac{a_j}{b_j + z}. \tag{11}$$

P_n and Q_n represents partial fractions:

$$\frac{P_n}{Q_n} = S_n(0), \qquad \frac{P_{n-1}}{Q_{n-1}} = S_n(\infty). \tag{12}$$

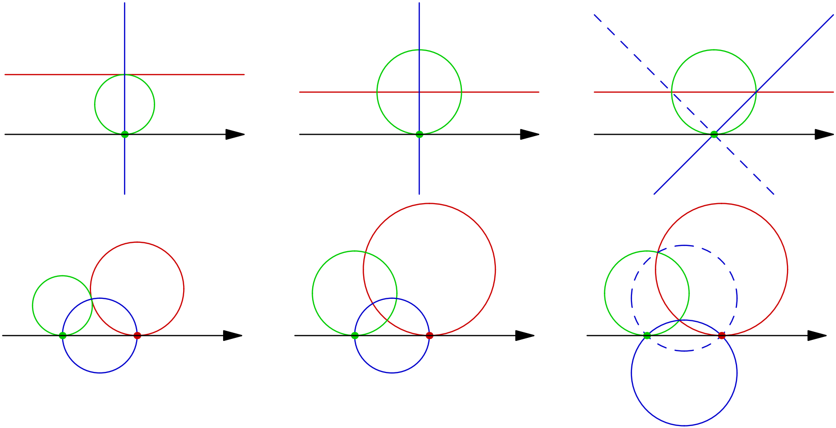
Lemma 1.

The cycles $(0, 0, 1, m)$ *(* $(k, 0, 1, 0)$ *) are the only cycles, such that their images under the Möbius transformation* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *are independent from the column* $\begin{pmatrix} b \\ d \end{pmatrix}$ *(*$\begin{pmatrix} a \\ c \end{pmatrix}$*). The image associated to the column* $\begin{pmatrix} a \\ c \end{pmatrix}$ *(*$\begin{pmatrix} b \\ d \end{pmatrix}$ *) is the horocycle, which touches the real line at* $\frac{a}{c}$ *(*$\frac{b}{d}$ *).*

A continued fraction is described by an infinite ensemble of cycles (C_k):

1. All C_k are touching the real line (i.e. are *horocycles*);
2. (C_1) is a horizontal line passing through $(0, 1)$;
3. C_{k+1} is tangent to C_k for all $k > 1$.

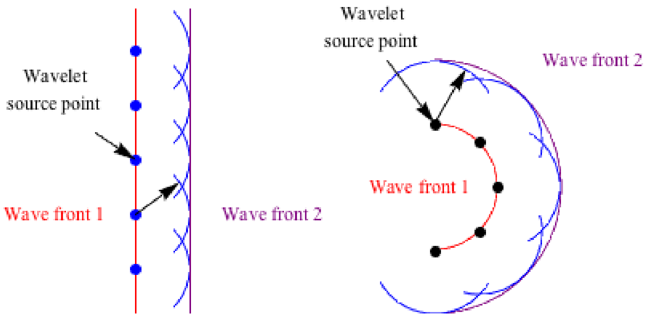
It was extended[8] to similar ensembles to treat convergence.



[8]Kisil, "Remark on Continued Fractions, Möbius Transformations and Cycles", 2016.

Spherical waves and their envelops

A physical example of an infinite ensemble: the representation of an arbitrary wave as the envelope of a continuous family of spherical waves.[9] A finite subset of spheres can be used as an approximation.



Further ideas of physical applications of FLT-invariant ensembles.[10]

[9] Bateman, *The mathematical analysis of electrical and optical wave-motion on the basis of Maxwell's equations*, 1955.

[10] Kastrup, "On the Advancements of Conformal Transformations and Their Associated Symmetries in Geometry and Theoretical Physics", 2008.

Extend Möbius–Lie Geometry

Definition 1.

The *extend Möbius–Lie geometry* considers ensembles of cycles interconnected through FLT-invariant relations.

Naturally, "old" objects—cycles—are represented by simplest one-element ensembles without any relation.

Conceptual foundations[11] of such extension and demonstrates its practical implementation as a CPP library **figure**. Interestingly, the development of this library shaped the general approach, which leads to specific realisations.[12][13][14]

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[11] Kisil, "An Extension of Lie Spheres Geometry with Conformal Ensembles of Cycles and Its Implementation in a `GiNaC` Library", 2014–2018.

[12] Kisil, "Poincaré Extension of Möbius Transformations", 2017.

[13] Kisil, "Remark on Continued Fractions, Möbius Transformations and Cycles", 2016.

[14] Kisil and Reid, "Conformal Parametrisation of Loxodromes by Triples of Circles", 2018.

Software implementation

The library **figure** (licensed under GNU GPLv3[15]) manipulates ensembles of cycles (quadrics) interrelated by certain FLT-invariant geometric conditions. The code is build on top of the previous library **cycle**,[16] which manipulates individual cycles within the GiNaC[17] computer algebra system.

It is important that both libraries are capable to work in spaces of any dimensionality and metrics with an arbitrary signatures: Euclidean, Minkowski and even degenerate. Parameters of objects can be symbolic or numeric, the latter admit calculations with exact or approximate arithmetic. Drawing routines work with any (elliptic, parabolic or hyperbolic) metric in two dimensions and the euclidean metric in three dimensions.

[15] GNU, *General Public License (GPL)*, 2007.

[16] Kisil, "Fillmore-Springer-Cnops Construction Implemented in GiNaC", 2007; Kisil, *Geometry of Möbius Transformations: Elliptic, Parabolic and Hyperbolic Actions of* $SL_2(\mathbf{R})$, 2012; Kisil, "Erlangen Program at Large–0: Starting with the Group $SL_2(\mathbf{R})$", 2007.

[17] Bauer, Frink, and Kreckel, "Introduction to the GiNaC Framework for Symbolic Computation within the C++ Programming Language", 2002.

Software implementation: Illustration

Thinking an ensemble as a graph: the library **cycle** deals with individual vertices (cycles), while **figure** considers edges (relations between pairs of cycles) and the whole graph.

The library **figure** reminds compass-and-straightedge constructions: new lines or circles are added to a drawing one-by-one through relations to already presented objects (the line through two points, the intersection point or the circle with given centre and a point).

```
F=figure()
a=F.add_cycle(cycle2D(1,[0,0],-1),"a")
l=symbol("l")
C=symbol("C")
F.add_cycle_rel([is_tangent_i(a),is_orthogonal(F.get_infinity()),\
    only_reals(l)],l)
F.add_cycle_rel([is_orthogonal(C),is_orthogonal(a),is_orthogonal(l),\
    only_reals(C)],C)
r=F.add_cycle_rel([is_orthogonal(C),is_orthogonal(a)],"r")
Res=F.check_rel(l,r,"cycle_orthogonal")
for i in range(len(Res)):
    print "Tangent and radius are orthogonal: %s" %\
    bool(Res[i].subs(pow(cos(wild(0)),2)==1-pow(sin(wild(0)),2))\
    .normal())
```

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Thank you for your attention! $\times 5$