# The Many Faces of Elastica

Ivaïlo Mladenov Mariana Hadzilazova

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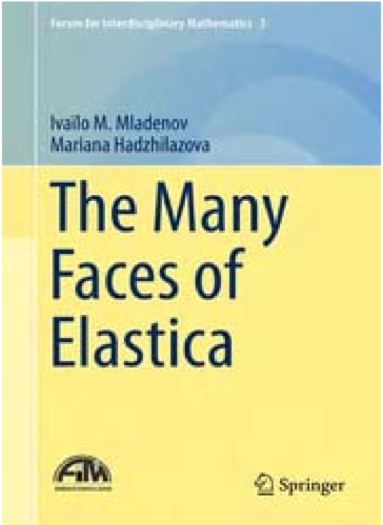
Ivailo M. Mladenov
Mariana Hadzhilazova
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# Chapter 2
# Planar Curves Whose Curvature Depends Only on the Distance from a Fixed Point

# The Moving Frame Associated With a Plane Curve

Here we consider in some details the cases in which the function in question is either proportional or inversely proportional to the distance from the origin. Let us start with the first case, namely

$$\kappa = \sigma r, \qquad r = |\mathbf{x}| = \sqrt{x^2 + z^2} \tag{1}$$

where $x, z$ are the Cartesian coordinates in the plane $XOZ$, which have to be considered as functions of the arc-length parameter $s$, and $\sigma$ is assumed to be a positive real constant.
If $\theta(s)$ denotes the slope of the tangent to the curve with respect to the $OX$ axis one has the following geometrical relations

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(s), \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta(s), \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \sin\theta(s) \tag{2}$$

# The Moving Frame Associated With a Plane Curve

which can be deduced also from the Frenet-Serret equations

$$\frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s} = \mathbf{T}(s), \qquad \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \kappa\mathbf{N}, \qquad \frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} \tag{3}$$

where **T** and **N** are are respectively the tangent and the normal vectors to the curve, and $s$ is the natural parameter along it. Combining (1) and (2) we get

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \kappa(r) \tag{4}$$

which is still a quite unpromising equation. We will proceed (as suggested but not pursued in Singer [1999]) by going to the co-moving frame $(\mathbf{T}, \mathbf{N})$ associated with the curve

# The Moving Frame Associated With a Plane Curve

$$\mathbf{x} = \xi \mathbf{T} + \eta \mathbf{N} \tag{5}$$

and accordingly the Frenet-Serret equations (3) take the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \dot{\xi} = \kappa\eta + 1, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}s} = \dot{\eta} = -\kappa\xi. \tag{6}$$

# Integration

Multiplying the first equation in (6) by $\xi$, the second one by $\eta$ and summing up the so obtained expressions, we find that

$$\xi = r\dot{r} \tag{7}$$

in which the dot means a differentiation with respect to the arc-length parameter. Substituting this expression back into the second equation of (6) and integrating we obtain

# Integration

$$\eta = -\int \kappa(r) r \mathrm{d}r + c \tag{8}$$

where $c$ is the integration constant. One should notice, however (cf. equation (5)), that the coordinates in the moving frame are not entirely independent but obey the constraint

$$\xi^2 + \eta^2 = r^2 \tag{9}$$

which in view of equations (7) and (8) presents an ordinary differential equation for the radial coordinate $r$.

# Bernoulli's Lemniscates

This curve is a special case (when $a \equiv c$) of the Cassinian ovals (see Mladenov [2000] ), defined by the equation

$$(x^2 + z^2)^2 - 2a^2(z^2 - x^2) + a^4 - c^4 = 0 \tag{10}$$

and has a curvature (which can be found using formula (**??**)), that is linear in $r$. Inserting $\kappa = \sigma r$ into equation (8) produces

$$\eta = -\frac{\sigma r^3}{3} \tag{11}$$

# Bernoulli's Lemniscates

(the integration constant is taken to be zero) and the scheme from the previous section leads to the equation

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1 - \frac{\sigma^2 r^4}{9}}\,. \qquad (12)$$

Its integration is immediate and gives

$$r = \sqrt{\frac{3}{\sigma}}\,\mathrm{cn}(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}) \qquad (13)$$

where $\mathrm{cn}(u, k)$ denotes one of the Jacobian elliptic functions in which the first slot is occupied by its argument and the second one by the so called elliptic modulus (a real number between zero and one).

# Bernoulli's Lemniscates

Substituting this solution into equations (7) and (8) has as a result the coordinates of the lemniscate in the moving frame

$$\begin{aligned} \xi &= -\sqrt{\frac{6}{\sigma}}\mathrm{cn}\Big(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\Big)\mathrm{dn}\Big(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\Big)\mathrm{sn}\Big(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\Big) \\ \eta &= -\sqrt{\frac{3}{\sigma}}\mathrm{cn}^3\Big(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}}\Big). \end{aligned} \tag{14}$$

With respect to the fixed one these functions give a new curve which we will call the co-lemniscate.

# Bernoulli's Lemniscates

Written in terms of its components equation (5) tells us that the lemniscate coordinates $x, z$ are obtained from those of the co-lemniscate $\xi, \eta$ via a plane rotation specified by the slope angle $\theta$, i.e.,

$$x = \xi \cos\theta - \eta \sin\theta, \qquad z = \xi \sin\theta + \eta \cos\theta. \tag{15}$$

Obviously, what remains to be done is to find $\theta$, and this can be obtained via an integration of the first equation in (2). In this way we obtain

$$\begin{aligned} \theta &= 3\arccos(\mathrm{dn}(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}})) \\ &= 3\arcsin(\tilde{k}\,\mathrm{sn}(\sqrt{\frac{2\sigma}{3}}s, \frac{1}{\sqrt{2}})), \qquad \tilde{k} = \sqrt{1-k^2}. \end{aligned} \tag{16}$$

# Bernoulli's Lemniscates

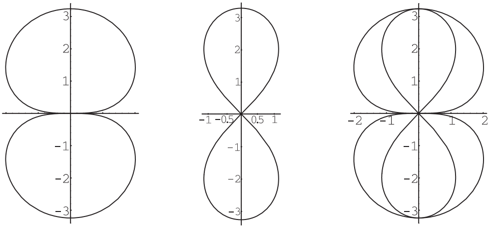


**Fig. 2.1** The Bernoullian co-lemniscate (left), Bernoulli's lemniscate (middle) and both of them (right) drawn via formulas (2.14) and (2.20) with $\sigma = 3.5$.

# Sturmian Spirals

By their very definition (cf. Zwickker [1963]) these plane curves possess the property that at each point their curvature radius $\mathcal{R}$ coincides with the distance $r$ from the origin. Formulated in curvature terms this means that their curvature $\kappa$ is given by formula (1), in which $\sigma \equiv 1$. Applying the scheme from Section ??, one finds easily that

$$\eta = -r + c \tag{17}$$

and

$$\dot{r} = \frac{\sqrt{2cr - c^2}}{r}, \qquad c > 0. \tag{18}$$

It is convenient to perform the integration of the above equation by switching to a new independent variable $t$ defined by the equation

$$\frac{\mathrm{d}s}{\mathrm{d}t} = r. \tag{19}$$

# Sturmian Spirals

This leads to the following results

$$r = \frac{c}{2}\left(t^2 + 1\right), \qquad \xi = ct, \qquad \eta = \frac{c}{2}\left(1 - t^2\right). \tag{20}$$

Integration of the first equation in (2) gives us additionally that the new parameter $t$ coincides (up to a real constant) with the slope angle, i.e.,

$$\theta = t. \tag{21}$$

By rewriting equation (5) in its components one has also the relations

$$x = \xi \cos t - \eta \sin t, \qquad z = \eta \cos t + \xi \sin t \tag{22}$$

# Sturmian Spirals

which combined with the above findings provides the sought-for parametrization of the Sturmian spirals

$$x = c\,(t\,\cos t + \frac{t^2-1}{2}\,\sin t), \quad z = c\,(\frac{1-t^2}{2}\,\cos t + t\,\sin t). \tag{23}$$

Making use of the above formulas one easily finds also the arc-length as a function of the parameter $t$, i.e.,

$$s = \frac{c}{2}(\frac{t^3}{3} + t). \tag{24}$$

By exchanging the numerical parameter $c$ for $2\rho$ and taking into account the fundamental relation $\mathcal{R} \equiv r$ (for this curve), the above formula can be written into the form, which is nothing else but the intrinsic equation (in Cesáro form) of the Sturm spiral

$$s = \frac{(\mathcal{R} + 2\rho)}{3}\sqrt{\frac{\mathcal{R}-\rho}{\rho}}\,. \tag{25}$$

# Generalized Sturm Spirals (The Case When $\sigma > 1$)

Due to the restriction on the allowed values of $\sigma$ one can consider as well the other two obvious possibilities, $\sigma > 1$ and $0 < \sigma < 1$, which have to be viewed as a generalization of the ordinary Sturmian spirals.

**The Case When $\sigma > 1$**

Here we will just outline the main ingredients of the derivation following again the scheme described before, starting with the equations

$$\eta = -\sigma r + c \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\sqrt{(1-\sigma^2)r^2 + 2c\sigma r - c^2}}{r}. \quad (26)$$

# Generalized Sturm Spirals (The Case When $\sigma > 1$)

One easily concludes that the expression under the radical on the right-hand side is positive provided that $c > 0$ and $r$ belongs either to a finite or infinite interval, i.e.,

$$\frac{c}{\sigma+1} \leq r \leq \frac{c}{\sigma-1} \quad \text{and} \quad \sigma > 1 \quad \text{or} \quad r > \frac{c}{\sigma+1} \quad \text{and} \quad \sigma < 1. \tag{27}$$

As the subsection title suggests our immediate task is to consider the first of the possibilities presented above. Exchanging as before the arc-length parameter (cf. equation (19)) with $t$, leads to the formula

$$r = \frac{c}{\sigma^2-1}(\sigma + \sin\sqrt{\sigma^2-1}\,t), \quad t \in [-\frac{\pi}{2\sqrt{\sigma^2-1}}, \frac{\pi}{2\sqrt{\sigma^2-1}}] \tag{28}$$

by which we find also

$$\xi = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{\sqrt{\sigma^2-1}}\cos\sqrt{\sigma^2-1}\,t, \qquad \theta = \sigma\, t. \tag{29}$$

# Generalized Sturm Spirals (The Case When $\sigma > 1$)

The combination of the above results with those from equation (28), the first equation in (26) and the general relations (22) gives

$$\begin{aligned} x &= c\left(\frac{\cos\sqrt{\sigma^2-1}\,t\,\cos\sigma t}{\sqrt{\sigma^2-1}} + \frac{(\sigma\sin\sqrt{\sigma^2-1}\,t+1)\sin\sigma t}{\sigma^2-1}\right) \\ z &= c\left(\frac{\cos\sqrt{\sigma^2-1}\,t\,\sin\sigma t}{\sqrt{\sigma^2-1}} - \frac{\left(\sigma\sin\sqrt{\sigma^2-1}\,t+1\right)\cos\sigma t}{\sigma^2-1}\right). \end{aligned} \tag{30}$$

# Generalized Sturm Spirals (The Case When $\sigma > 1$)

The expressions for the arc-length and the intrinsic equation in this case are

$$s = \frac{c}{\sigma^2 - 1}(\sigma t - \frac{\cos\sqrt{\sigma^2 - 1}\,t}{\sqrt{\sigma^2 - 1}} + \frac{\sigma\pi}{2\sqrt{\sigma^2 - 1}}) \tag{31}$$

and

$$\begin{aligned} s \;&=\; \frac{c}{\sigma^2 - 1}\left(\frac{\sigma}{\sqrt{\sigma^2 - 1}}\arcsin(\frac{\sigma}{c}\left((\sigma^2 - 1)\mathcal{R} - c\right))\right. \\ &\left. - \frac{1}{c}\sqrt{\sigma^2(1 - \sigma^2)\mathcal{R}^2 + 2c\sigma^2\mathcal{R} - c^2} + \frac{\sigma\pi}{2\sqrt{\sigma^2 - 1}}\right) \end{aligned} \tag{32}$$

where in the derivation of the last equation we have used the defining relation for the spiral which in this case states that $r = \sigma\mathcal{R}$.

# Generalized Sturm Spirals (The Case When $\sigma > 1$)

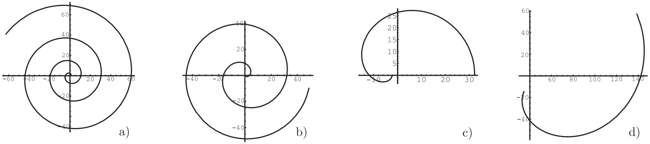


**Fig. 2.2** a) The standard Sturmian spiral generated by (2.36) and $c = 0.25$, and the generalized Sturmian spirals drawn via formulas in (2.43) with the following set of the parameters, b) $c = 1$, $\sigma = 1.02$, c) $c = 5$, $\sigma = \frac{2}{\sqrt{3}}$ d) $c = 100$, $\sigma = 5/3$.

# Generalized Sturm Spirals. (The Case When $0 < \sigma < 1$)

The first steps in the scheme amount to

$$\eta = -\sigma r + c, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\sqrt{(1-\sigma^2)r^2 + 2c\sigma r - c^2}}{r} \tag{33}$$

but one should keep in mind that now $\sigma < 1$ and $r > \frac{c}{\sigma+1}\cdot$ It turns out also more convenient to perform the integration of the equation on the right-hand side in (33) by introducing the parameter $\tau$ via the equation

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = r^2 \tag{34}$$

# Generalized Sturm Spirals. (The Case When $0 < \sigma < 1$)

which produces

$$r = \frac{c}{\sigma - \sin c\tau}, \qquad \tau \in [-\frac{\pi}{2\,c}, \frac{\arcsin \sigma}{c}] \tag{35}$$

and

$$\begin{aligned} \xi &= -\frac{c \cos c\tau}{\sigma - \sin c\tau}, \qquad \eta = -\frac{c \sin c\tau}{\sigma - \sin c\tau} \\ \theta(\tau) &= \frac{\sigma}{\sqrt{1-\sigma^2}} \ln \frac{\sigma \tan \frac{c\tau}{2} - \sqrt{1-\sigma^2} - 1}{\sigma \tan \frac{c\tau}{2} + \sqrt{1-\sigma^2} - 1}\,. \end{aligned} \tag{36}$$

# Generalized Sturm Spirals. (The Case When $0 < \sigma < 1$)

Further, via equations (22) and (36) we obtain

$$x = \frac{c}{\sigma - \sin c\tau} \cos\left( c\tau - \frac{\sigma}{\sqrt{1-\sigma^2}} \ln \frac{\sigma \tan \frac{c\tau}{2} - \sqrt{1-\sigma^2} - 1}{\sigma \tan \frac{c\tau}{2} + \sqrt{1-\sigma^2} - 1} \right) \tag{37}$$

$$z = \frac{c}{\sigma - \sin c\tau} \sin\left( c\tau - \frac{\sigma}{\sqrt{1-\sigma^2}} \ln \frac{\sigma \tan \frac{c\tau}{2} - \sqrt{1-\sigma^2} - 1}{\sigma \tan \frac{c\tau}{2} + \sqrt{1-\sigma^2} - 1} \right)$$

and finally

$$s = \frac{c\sigma}{(1-\sigma^2)^{3/2}} \ln \left( \frac{\sigma + (1+\sqrt{1-\sigma^2}) \tan\left(\frac{1}{2}\arcsin\left(\frac{c}{r} - \sigma\right)\right)}{1 + \sqrt{1-\sigma^2} + \sigma \tan\left(\frac{1}{2}\arcsin\left(\frac{c}{r} - \sigma\right)\right)} \right) + \frac{\sqrt{(1-\sigma^2) r^2 + 2c\sigma r - c^2}}{1-\sigma^2}. \tag{38}$$

As before, one can easily obtain from the last expression the intrinsic equation of the curve by replacing $r$ with $\sigma\mathcal{R}$.

# Generalized Sturm Spirals. (The Sub-Case When $0 < \sigma < 1$ and $c = 0$)

Just for completeness we will consider the situation when the integration constant $c$ which appears in the previous subsection is zero. Obviously, the equations in (33) simplify to

$$\eta = -\sigma r, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1-\sigma^2}. \tag{39}$$

The integration of the second one is immediate and gives

$$r = \sqrt{1-\sigma^2}\, s + a \tag{40}$$

# Generalized Sturm Spirals. (The Sub-Case When $0 < \sigma < 1$ and $c = 0$)

where $a$ denotes a new integration constant that is necessarily positive. Following the scheme one ends with the results

$$\xi = \left(1 - \sigma^2\right) s + a\sqrt{1 - \sigma^2}, \qquad \eta = -\sigma \left(\sqrt{1 - \sigma^2}\, s + a\right) \quad (41)$$

and

$$\theta(s) = \frac{\sigma}{\sqrt{1 - \sigma^2}} \ln(\sqrt{1 - \sigma^2}\, s + a) \quad (42)$$

which allow us to write down the explicit parametrization of the corresponding spiral

$$\begin{aligned} x &= ((1 - \sigma^2)s + a\sqrt{1 - \sigma^2}) \cos\theta(s) + \sigma(\sqrt{1 - \sigma^2}\, s + a) \sin\theta(s) \\ z &= ((1 - \sigma^2)s + a\sqrt{1 - \sigma^2}) \sin\theta(s) - \sigma(\sqrt{1 - \sigma^2}\, s + a) \cos\theta(s). \end{aligned} \quad (43)$$

# Generalized Sturm Spirals. (The Sub-Case When $0 < \sigma < 1$ and $c = 0$)

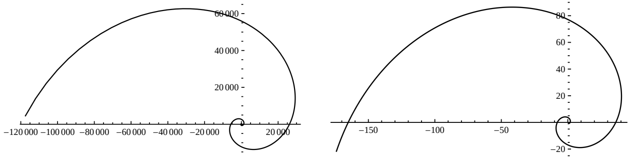


**Fig. 2.3** Sturmian spirals generated by formula (2.50) and the constants $\sigma = 0.9$, $c = 1$ (left) and formula (2.56) with $\sigma = 0.9$ and $a = 1$ (right).

## Serret Curves

We should note that recently Lipkovski [1996] was able to prove that all Serret curves $\mathcal{S}_n$ are rational ones, i.e., they admit rational parametrizations.

Going back to the original Serret [1845a] writings one can find a formula for the curvature of $\mathcal{S}_n$ in the form

$$\kappa(r) = \frac{3r}{2\sqrt{n(n+1)}} - \frac{2n+1}{2\sqrt{n(n+1)}r} \tag{44}$$

which depends solely on the radial coordinate $r$. This will be used in the next section to generate them by following the original construction of Mladenov *et al* [2010], [2012].

# Generalized Serret Curves

The expression for the curvature of Serret's curves (44) suggests immediately a generalization of the form

$$\kappa(r) = 3\lambda r - \frac{\sigma}{r}, \qquad \lambda > 0, \qquad \sigma > 1. \tag{45}$$

Substitution of (45) in (8) produces

$$\eta = -\lambda r^3 + \sigma r \tag{46}$$

but one has to notice that the integration constant in (8) is taken to be zero. In these circumstances the differential equation

$$\dot{r}^2 = \frac{1}{r^2}\left(r^2 - \eta^2\right) \tag{47}$$

## Generalized Serret Curves

which follows from (9) reduces to the equation

$$\frac{\mathrm{d}r}{\sqrt{(a^2 - r^2)(r^2 - c^2)}} = \lambda \mathrm{d}s \tag{48}$$

in which the real parameters $a$ and $c$ are given by the formulas

$$a = \sqrt{\frac{\sigma + 1}{\lambda}}, \qquad c = \sqrt{\frac{\sigma - 1}{\lambda}}. \tag{49}$$

The integration of (48) can be performed in terms of the Jacobian elliptic function $\mathrm{dn}(\cdot, \cdot)$, namely

$$r(s) = a\mathrm{dn}(a\lambda s, k), \qquad k = \sqrt{\frac{2}{\sigma + 1}} \tag{50}$$

## Generalized Serret Curves

The next step in the scheme amounts to the evaluation of the integral

$$\theta(s) = \int \kappa(r(s))\mathrm{d}s \tag{51}$$

and this gives

$$\theta(s) = 3\mathrm{am}(\sqrt{\lambda(\sigma+1)}\, s, k) - \frac{\sigma}{\sqrt{\sigma^2-1}} \arccos \frac{\mathrm{cn}(\sqrt{\lambda(\sigma+1)}\ s, k)}{\mathrm{dn}(\sqrt{\lambda(\sigma+1)}\ s, k)} \tag{52}$$

where $\mathrm{am}(t, k)$ is the Jacobian amplitude function and $\mathrm{cn}(t, k) = \cos \mathrm{am}(t, k)$.
Having at out disposal (7), (46), (50) and (52) can enter into (15) and this gives the parametrization of the generalized Serret curves.

# Generalized Serret Curves

Obviously the parametrization of the classical Serret curves can be obtained by taking

$$\lambda = \frac{1}{2\sqrt{n(n+1)}}, \qquad \sigma = \frac{2n+1}{2\sqrt{n(n+1)}}, \qquad n \in \mathbb{N} \tag{53}$$

and in this case the slope angle turns out to be

$$\theta_n(s) = 3\mathrm{am}(\mu_n s, k_n) - (2n+1)\arccos\frac{\mathrm{cn}(\mu_n s, k_n)}{\mathrm{dn}(\mu_n s, k_n)} \tag{54}$$

where

$$\mu_n = \frac{1}{2}\sqrt{\frac{2\sqrt{n(n+1)}+2n+1}{n(n+1)}}, \qquad k_n = 2\sqrt{\frac{\sqrt{n(n+1)}}{2\sqrt{n(n+1)}+2n+1}}. \tag{55}$$

Several plots of both classical and generalized Serret's curves are presented in Fig. ?? and Fig. ??.

# Generalized Serret Curves

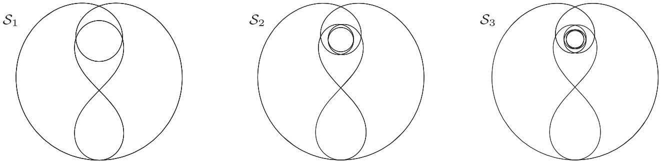


**Fig. 2.5** The classical Serret's curves $\mathcal{S}_1$ - left, $\mathcal{S}_2$ - middle and $\mathcal{S}_3$ - right for $n = 1, 2, 3$.

# Generalized Serret Curves

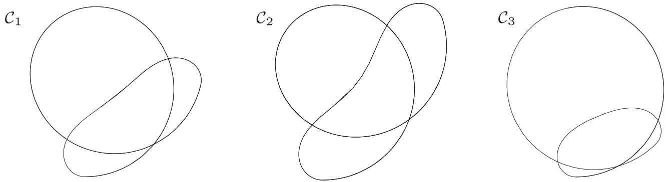


**Fig. 2.6** Three examples of the generalized Serret's curves $\mathcal{C}_1$ (left), $\mathcal{C}_2$ (middle) and $\mathcal{C}_3$ (right) generated respectively with parameter sets $\lambda = 1/3, \sigma = 7/5$, $\lambda = 4/3, \sigma = 9/7$ and $\lambda = 1/7, \sigma = 5/3$.

# Cassinian Ovals

This remarkable plane curve is defined as the geometrical locus of the points in the plane for which the product of the distances from two fixed points $\mathbf{F}_1$ and $\mathbf{F}_2$ is a constant, which will be denoted with $c^2$, and *the distance* $(\mathbf{F}_1, \mathbf{F}_2)$ between $\mathbf{F}_1$ and $\mathbf{F}_2$ is also a constant denoted with $2a$. In the $XOZ$ plane the Cassinian ovals are given by the equation (an alternative form is (10))

$$(x^2 + z^2 + a^2)^2 - 4a^2x^2 = c^4. \tag{56}$$

It is clear that these curves are symmetrical with respect to both coordinate axes. Their shapes depend on the precise relationship of the geometrical parameters $a$ and $c$.

# Cassinian Ovals

From now on we will consider the case when $a < c < a\sqrt{2}$ (this is the case *3* in Fig. **??**). For $c \geqq a\sqrt{2}$ we have ellipse like figures illustrated by curves numbered as *4* and *5*, and when $c = a$ the curve is given by the equation

$$(x^2 + z^2)^2 = 2a^2(x^2 - z^2) \tag{57}$$

which is nothing else than the Bernoullian lemniscate reproduced here as curve *2*. Finally in the case when $a > c$ the curves reduce to two disjoint ovals (these ovals are depicted as curves *1* in Fig. **??**).

# Cassinian Ovals

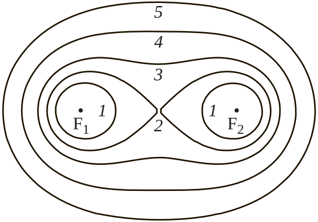


**Fig. 2.7** Cassinian ovals drawn with different values of dimensionless ratio $\varepsilon = a/c$.

# Cassinian Ovals

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$$(x^2 + z^2)^2 = 2a^2(x^2 - z^2) \tag{58}$$

which is nothing else than the Bernoullian lemniscate reproduced here as curve *2*. Finally in the case when $a > c$ the curves reduce to two disjoint ovals (these ovals are depicted as curves *1* in Fig. **??**).

# Cassinian Ovals

Cassini has proposed the fourth degree curves (56) in an attempt to describe properly the planetary motions in the solar system. The equations (56) and (58) describe concrete algebraic curves. The meaning of the last notion is that the rectangular coordinates $x, z$ of the points on the curve $C$ in the plane satisfy an algebraic equation

$$F(x, z) = 0 \tag{59}$$

where $F(x, z)$ is a polynomial function in its variables.
Following the tradition established by Canham [1970], Deuling & Helfrich [1976], Funaki [1955] and Vayo [1983] the Cassinian ovals can be considered as a model of red blood cells. For more detail see Angelov & Mladenov [2000] and Hadzhilazova *et al* [2011].

*Thank you for attention!*