The Many Faces of Elastica

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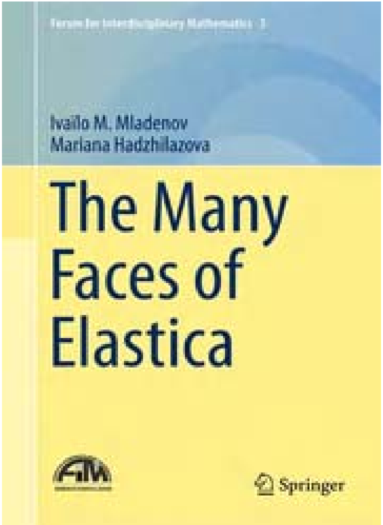
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The Many Faces of Elastica
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Chapter 4
Surface Tension and Equilibrium

Tensions and Geometry

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}(\sigma_m r) + \sigma_c \sin\theta - wr\cos\theta \\ \frac{\partial}{\partial s}(\sigma_m r) &= -\sigma_c \sin\theta + wr\cos\theta. \end{aligned} \tag{1}$$

Dotting with $\bar{\mathbf{n}}$ gives (using $\partial \mathbf{t}/\partial s = -\theta' \bar{\mathbf{n}}$)

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}(\sigma_m r \mathbf{t}) \cdot \bar{\mathbf{n}} - \sigma_c \mathbf{e}_1 \cdot \bar{\mathbf{n}} + pr\bar{\mathbf{n}} \cdot \bar{\mathbf{n}} - wr\mathbf{k} \cdot \bar{\mathbf{n}} \\ 0 &= -\sigma_m r\theta' - \sigma_c \cos\theta + pr - wr\sin\theta \\ \sigma_m r\theta' &= pr - \sigma_c \cos\theta - wr\sin\theta. \end{aligned} \tag{2}$$

Mylar Balloon and Elastic Curves

This is another case in which the system of equations (1) and (2) can be solved up to the very end. Assuming that $w(s) = \sigma_c = 0$ and that the hydrostatic pressure $p(s) = p_o$ is a constant (and non-zero), this system reduces to the equations

$$(\sigma_m(s) r(s)) \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = p_o r(s) \tag{3}$$

$$\frac{\mathrm{d}(\sigma_m(s) r(s))}{\mathrm{d}s} = 0. \tag{4}$$

Mylar Balloon and Elastic Curves

The second equation above tells us that $\sigma_m(s)r(s)$ is a constant quantity and therefore we can introduce the meridional stress resultant $\mathring{\sigma}$ on the equator of the balloon, i.e., the points for which $r(s) = a$, $z(s) = 0$ and rewrite the above integral in the form

$$\sigma_m(s) = \frac{a\mathring{\sigma}}{r(s)}. \tag{5}$$

This allows us also to rewrite the first equation (3) as

$$\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \mathring{p}r(s), \qquad \mathring{p} = \frac{p_o}{a\mathring{\sigma}} > 0. \tag{6}$$

Mylar Balloon and Elastic Curves

If we combine this equation with

$$r'(s) = -\sin\theta(s), \qquad z'(s) = \cos\theta(s) \tag{7}$$

we get the following geometrical relation

$$r^2(s) = \frac{2}{\mathring{p}} \cos\theta(s). \tag{8}$$

This last relation, as we shall see, characterizes uniquely the surface in question. Let us start with solving (8) for $r(s)$. After that we replace the result in (6) and in this way obtain a differential equation with separated variables

$$\frac{\mathrm{d}\theta}{\sqrt{\cos\theta}} = \sqrt{2\mathring{p}}\,\mathrm{d}s. \tag{9}$$

Mylar Balloon and Elastic Curves

Next we introduce $\eta = \sin\theta$, which transforms the left hand side into

$$\frac{\mathrm{d}\eta}{\sqrt{\eta(1-\eta^2)}}$$

and this suggest a new change $\eta = \zeta^2$ of the independent variable η, which gives

$$\frac{\mathrm{d}\eta}{\sqrt{\eta(1-\eta^2)}} = \frac{2\mathrm{d}\zeta}{\sqrt{1-\zeta^4}} = \frac{2\mathrm{d}\zeta}{\sqrt{(1+\zeta^2)(1-\zeta^2)}}.$$

By all these changes equation (9) reduces to the form

$$\frac{2\mathrm{d}\zeta}{\sqrt{(1+\zeta^2)(1-\zeta^2)}} = \sqrt{2\mathring{p}}\,\mathrm{d}s. \qquad (10)$$

Mylar Balloon and Elastic Curves

However, this is a standard elliptic integral (see Jahnke *et al* [1960]), which can be inverted directly in Jacobi's eliptic functions cn, and we have

$$\zeta(s) = -\mathrm{cn}\left(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}\right).$$

As a consequence we have also

$$\sin\theta(s) = \frac{\mathring{p}}{2}r^2(s) = \zeta^2(s) = \mathrm{cn}^2\left(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}\right) \tag{11}$$

and therefore

$$r(s) = \sqrt{\frac{2}{\mathring{p}}}\mathrm{cn}\left(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}\right). \tag{12}$$

Mylar Balloon and Elastic Curves

In order to find $z(s)$ we make use of

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\tan\theta = -\frac{\mathring{p}r^2 + 2C}{\sqrt{(c^2 - r^2)(r^2 - a^2)}} \tag{13}$$

and (11) which lead to

$$\frac{\mathrm{d}z(s)}{\mathrm{d}s} = -\mathrm{cn}^2\left(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}\right). \tag{14}$$

Details about the integration of the above equation can be found in Hadzhilazova & Mladenov [2006], and the result is

$$z(s) = -\frac{2}{\sqrt{\mathring{p}}}\left[E\left(\mathrm{am}\left(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right) - \frac{1}{2}F\left(\mathrm{am}\left(\sqrt{\mathring{p}}s, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\right)\right]. \tag{15}$$

Mylar Balloon and Elastic Curves

If we compare the obtained parametrization of the profile curve $(r(s), z(s))$ provided by (12) and (15) with that one in Mladenov & Oprea [2003] we can easily conclude that we are dealing here with the Mylar balloon. The profile curve and the surface generated by them are shown in Fig. 1 and Fig. 2.

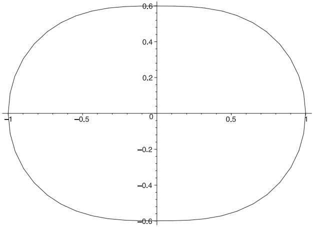

Figure: The profile of the mylar balloon in XOZ plane.

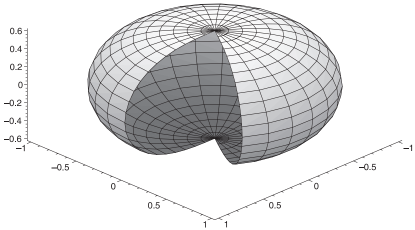

Figure: An open part of the mylar balloon.

Mylar Balloon and Elastic Curves.

For commercial purposes the just mentioned Mylar balloon is fabricated from two circular disks of mylar, sewing them along their boundaries and then inflating. Surprisingly enough, these balloons are not spherical as one naively might expect from the well-known fact that the sphere possesses the maximal volume for a given surface area. An experimental fact like this suggests a mathematical problem regarding the exact shape of the balloon when it is fully inflated.

This problem was first spelled out by Paulsen [1994] in a variational setting while here we have provided in fact its non-variational characterization. One should mention also the remarkable scale invariance (i.e., independence of the actual size) of the thickness to diameter ratio of the inflated balloon which turns out to be with a good approximation equal to 0.599 (see Mladenov [2002a]).

Mylar Balloon and Elastic Curves.

Another important fact about this surface is the very simple expression for its area given by the formula $\mathcal{A} = \pi^2 a^2$, where a is the radius of the inflated balloon.In some sense all these nice properties are due to the remarkable property which specifies uniquely the mylar balloon as the only surface of revolution for which the principal curvatures k_μ and k_π obey to the equation

$$k_\mu = 2k_\pi. \tag{16}$$

As has been noticed by Gibbons [2006] this (Weingarten) property can be derived within membrane approach as well by rewriting (8) in the form

$$-\mathring{p}r(s) = -2\frac{\cos\theta(s)}{r(s)}. \tag{17}$$

Mylar Balloon and Elastic Curves.

Taking into account (6) along with the definitions of the principal curvatures given in

$$k_{\mu} = \frac{g''h' - g'h''}{(g'^2 + h'^2)^{3/2}}, \qquad k_{\pi} = \frac{g'}{h(g'^2 + h'^2)^{1/2}} \cdot \tag{18}$$

amounts directly to the equality (16).

Bending Energy

When one consider the elastic properties of the materials the main question is: What is the energy needed to bend a rod in the plane? According to the theory of elasticity (see Love [1944]) the energy is proportional to the integral of the squared curvature along its length

$$U = \int_0^L \frac{EI}{\mathcal{R}^2} \mathrm{d}s \tag{19}$$

where U is the bending energy, E is the Young's module of elasticity, I is the moment about the neutral axis, $\mathcal{R}$ is the curvature radius of the neutral axis, and L is its lenght.

Mylar Balloon and Elastic Curves. Bending Energy

Its study was initiated by James Bernoulli in 1691 who had tried to develop this model by using the available mechanics, geometry and variational calculus and this approach has been continued in Euler's book [1744], which lays down the basis of the modern variational calculus.

There he examines the problem of finding the shape of an elastic ribbon with a fixed length L, which connects two points in the plane at which it has fixed tangents (see Fig. 3). We assume that the origin O of the coordinate system XOZ coincides with one of the points and that the other has coordinates (x, z).

Mylar Balloon and Elastic Curves. Bending Energy

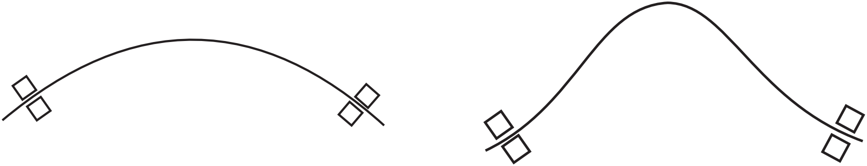

Figure: Some possible positions of the ribbon when it is free - on the left and with a fixed length - on the right.

Mylar Balloon and Elastic Curves. Bending Energy

Now our task is to find the minimum of the functional

$$J_0 = \int_0^L \kappa^2(s) \tag{20}$$

under the condition of a fixed length

$$\int_0^L \mathrm{d}s = L. \tag{21}$$

Bernoulli and Euler make use of the equation (**??**) and respectively J_0 reads

$$\int \frac{\ddot{z}^2(x)}{(1+\dot{z}^2)^{5/2}}\mathrm{d}x, \qquad \dot{z} = \frac{\mathrm{d}z}{\mathrm{d}x}, \qquad \ddot{z} = \frac{\mathrm{d}^2 z}{\mathrm{d}x^2}. \tag{22}$$

Mylar Balloon and Elastic Curves. Bending Energy

Despite that in this case the Euler-Lagrange equation is of fourth order, Euler managed to integrate it three times and to find the first order equation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{a^2 - c^2 + x^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}, \qquad a, c \in \mathbb{R}. \tag{23}$$

From this point on his considerations were qualitative (the elliptic fuctions were not invented yet!) and the solutions of (23) were classified according to the different values of the real parameter

$$m = \frac{a^2}{2c^2}. \tag{24}$$

In the next section we will expose the above result by following the original Euler's method.

Mylar Balloon and Elastic Curves.
Original Formulation and Treatment of the Problem About Elastic Curves

In this section we will consider the original method of Euler, which aims the integration of the equation that describes the elasticas. As mentioned before it is a nonlinear fourth order ordinary differential equation.

Let us start by writing the arc length $\mathrm{d}s$ in the form

$$\mathrm{d}s = (\mathrm{d}x^2 + \mathrm{d}z^2)^{1/2} = (\mathrm{d}x^2 + \dot{z}^2\mathrm{d}x^2)^{1/2} = (1 + \dot{z}^2)^{1/2}\mathrm{d}x \quad (25)$$

and for the full functional J we can write respectively

$$J = \int \frac{\ddot{z}^2\mathrm{d}x}{(1 + \dot{z}^2)^{5/2}} + \lambda \int (1 + \dot{z}^2)^{1/2}\mathrm{d}x.$$

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

The Euler-Lagrange equation is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial \ddot{z}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \dot{z}}\right) + \frac{\partial F}{\partial z} = 0$$

and in our case it is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(2\ddot{z}(1+\dot{z}^2)^{-5/2}) + \frac{\mathrm{d}}{\mathrm{d}x}(5\ddot{z}^2\dot{z}(1+\dot{z}^2)^{-7/2} - \lambda\dot{z}(1+\dot{z}^2)^{-1/2}) = 0.$$

Integrating once the last equation we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(2\ddot{z}(1+\dot{z}^2)^{-5/2}) + 5\ddot{z}^2\dot{z}(1+\dot{z}^2)^{-7/2} - \lambda\dot{z}(1+\dot{z}^2)^{-1/2} = A$$

where A is an integration constant. Performing the differentiation in the above formula we have

$$2\dddot{z}(1+\dot{z}^2)^{-5/2} - 5\ddot{z}^2\dot{z}(1+\dot{z}^2)^{-7/2} + \lambda\dot{z}(1+\dot{z}^2)^{-1/2} = A. \quad (26)$$

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

In this equation we can rewrite $2\dddot{z}$ which appears in the first term as

$$2\dddot{z} = 2\ddot{z}\frac{\mathrm{d}\ddot{z}}{\mathrm{d}\dot{z}} = \frac{\mathrm{d}\ddot{z}^2}{\mathrm{d}\dot{z}}$$

the second one as

$$\ddot{z}^2\frac{\mathrm{d}}{\mathrm{d}\dot{z}}(1+\dot{z}^2)^{-5/2}$$

and the third one as

$$-\lambda\frac{\mathrm{d}}{\mathrm{d}\dot{z}}(1+\dot{z}^2)^{1/2}.$$

Doing so the equation (26) transforms in to the form

$$(1+\dot{z}^2)^{-5/2}\frac{\mathrm{d}\ddot{z}^2}{\mathrm{d}\dot{z}} + \ddot{z}^2\frac{\mathrm{d}}{\mathrm{d}\dot{z}}(1+\dot{z}^2)^{-5/2} - \lambda\frac{\mathrm{d}}{\mathrm{d}\dot{z}}(1+\dot{z}^2)^{1/2} = A.$$

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

Now, we can unify the first two terms in the last equation to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\dot{z}}(\ddot{z}^2(1+\dot{z}^2)^{-5/2}) - \frac{\mathrm{d}}{\mathrm{d}\dot{z}}(\lambda(1+\dot{z}^2)^{1/2}) = A. \tag{27}$$

After a direct integration of (27) we get

$$\ddot{z}^2(1+\dot{z}^2)^{-5/2} = \lambda(1+\dot{z}^2)^{1/2} + A\dot{z} + B \tag{28}$$

where B is the new integration constant.
Solving the last equation for $\ddot{z}$ yields

$$\ddot{z} = \frac{\mathrm{d}\dot{z}}{\mathrm{d}x} = (1+\dot{z}^2)^{5/4}\left(\lambda(1+\dot{z}^2)^{1/2} + A\dot{z} + B\right)^{1/2}. \tag{29}$$

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

A crucial moment is the observation (due to Euler) that

$$\frac{\mathrm{d}}{\mathrm{d}\dot{z}}\left(\frac{2(\lambda(1+\dot{z}^2)^{1/2}+A\dot{z}+B)^{1/2})}{(1+\dot{z}^2)^{1/4}}\right)=\frac{A-B\dot{z}}{(1+\dot{z}^2)^{5/4}\left(\lambda(1+\dot{z}^2)^{1/2}+A\dot{z}+B\right)^{1/2}}$$

and taking into account (29) we end up with

$$\frac{\mathrm{d}}{\mathrm{d}\dot{z}}\left(\frac{2\left(\lambda(1+\dot{z}^2)^{1/2}+A\dot{z}+B\right)^{1/2})}{(1+\dot{z}^2)^{1/4}}\right)=(A-B\dot{z})\frac{\mathrm{d}x}{\mathrm{d}\dot{z}}. \qquad (30)$$

The integration of the above equation is immediate and produces

$$\frac{2(\lambda(1+\dot{z}^2)^{1/2}+A\dot{z}+B)^{1/2})}{(1+\dot{z}^2)^{1/4}}=Ax-Bz+C \qquad (31)$$

where C is the integration constant.

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

By making a special choice for the constants B and C, i.e., $B = C = 0$ and solving the so reduced equation (31) for $\dot{z}$ we obtain the equation

$$\dot{z}(x) = \frac{A^2x^2 - 4\lambda}{\sqrt{16A^2 - (A^2x^2 - 4\lambda)^2}}. \qquad (32)$$

At this stage it is convenient to introduce the real numbers a and c as new parameters via the relations

$$a^4 = \frac{16}{A^2} \qquad \text{and} \qquad c^2 - a^2 = \frac{4\lambda}{A^2} \qquad (33)$$

and in this way to convert (32) into

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{a^2 - c^2 + x^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}$$

which is exactly the Eulerian equation (23).

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

Alternatively, recalling the formula for the curvature

$$\kappa(x) = \frac{\ddot{z}}{(1+\dot{z}^2)^{3/2}}$$

we can transform equation (28) into the form

$$\kappa^2 = A\frac{\dot{z}}{\sqrt{1+\dot{z}^2}} + \frac{B}{\sqrt{1+\dot{z}^2}} + \lambda. \tag{34}$$

According to Fig. **??** one has $\dot{z} = \frac{\mathrm{d}z}{\mathrm{d}x} = \tan\psi$, where $\psi = \psi(x)$ is the angle between the tangent at this point and the positive direction of OX axis and hence the equation (34) becomes

$$\kappa^2 = A\sin\psi + B\cos\psi + \lambda.$$

Mylar Balloon and Elastic Curves.
Original Formulation and Treatment of the Problem About Elastic Curves

Differentiating this equality with respect to the arclength s (which we will denote with a prime) give us

$$2\kappa\kappa' = A\cos\psi\psi' - B\sin\psi\psi'. \tag{35}$$

By its very definition

$$\kappa = \frac{\mathrm{d}\psi}{\mathrm{d}s} = \psi'$$

and therefore we have the equation

$$2\kappa' = A\cos\psi - B\sin\psi \tag{36}$$

in which for the left hand side we can write

$$\kappa' = \frac{\mathrm{d}\kappa}{\mathrm{d}s} = \frac{\mathrm{d}\kappa}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{\sqrt{1+\dot{z}^2}}\frac{\mathrm{d}\kappa}{\mathrm{d}x} = \frac{\dot{\kappa}}{\sqrt{1+\dot{z}^2}}.$$

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

In Cartesian coordinates (36) becomes

$$\frac{2\dot{\kappa}}{\sqrt{1+\dot{z}^2}} = \frac{A}{\sqrt{1+\dot{z}^2}} - \frac{B\dot{z}}{\sqrt{1+\dot{z}^2}}$$

which obviously is equivalent to the equation

$$2\dot{\kappa} = A - B\dot{z}.$$

Integrating the above relation we get the equation of the Eulerian elasticas

$$\kappa(x,z) = \frac{A}{2}x - \frac{B}{2}z + C, \qquad C = \mathrm{const} \tag{37}$$

which tells us that elasticas are curves which curvatures are linear functions of the Cartesian coordinates.

Mylar Balloon and Elastic Curves. Original Formulation and Treatment of the Problem About Elastic Curves

If we take as the ordinate the line

$$Bx + Az + D = 0$$

and for abscissa the perpendicular one

$$Ax - Bz + 2C = 0$$

then the expression for the curvature of the elasticas (37) transforms into

$$\kappa(x) = \alpha x, \qquad \alpha \in \mathbb{R} \tag{38}$$

where we have used the same notation for the new quantities.
Remark. One can check the compatibility of the approaches presented above by evaluating the curvature using (23) which gives (modulo some calculations) the expected formula

$$\kappa(x) = \frac{2}{a^2} x. \tag{39}$$

Mylar Balloon and Elastic Curves.
Parametric Representation of Curvature of Elastica

According to the alternative formula $\kappa(s) = \sqrt{\mathbf{x}'' \cdot \mathbf{x}''} = (x_{ss}^2 + z_{ss}^2)^{1/2}$. for the curvature of plane curves, the Euler's elasticas problem reduces to the study of the functional

$$J = \int (\ddot{x}^2 + \ddot{z}^2 + \nu(s)(\dot{x}^2 + \dot{z}^2))\mathrm{d}s.$$

The over dot here means a differentiation with respect of the natural parameter. Respectively, the Euler-Lagrange's equations

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial F}{\partial \ddot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{x}} + \frac{\partial F}{\partial x} = 0 \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial F}{\partial \ddot{z}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{z}} + \frac{\partial F}{\partial z} = 0.$$

in this case reduces to the system

$$\frac{\mathrm{d}^2\ddot{x}}{\mathrm{d}s^2} - \frac{\mathrm{d}}{\mathrm{d}s}(\nu\dot{x}) = 0, \qquad \frac{\mathrm{d}^2\ddot{z}}{\mathrm{d}s^2} - \frac{\mathrm{d}}{\mathrm{d}s}(\nu\dot{z}) = 0$$

which can be rewritten in their equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dddot{x} - \nu\dot{x}) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}s}(\dddot{z} - \nu\dot{z}) = 0.$$

Mylar Balloon and Elastic Curves. Parametric Representation of Curvature of Elastica

From the formulas above we can obtain immediately the equations

$$\dddot{x} - \nu \dot{x} = A, \qquad \dddot{z} - \nu \dot{z} = B \tag{40}$$

where$A$ and B are some integration constants. Multiplying the first equation with $\dot{z}$, the second with $\dot{x}$ and subtracting the second result from the first one we obtain

$$\dddot{x}\,\dot{z} - \dot{x}\,\dddot{z} = A\dot{z} - B\dot{x}.$$

Further, integrating once more give us

$$\ddot{x}\dot{z} - \dot{x}\ddot{z} = Az - Bx + C.$$

The left hand side of the above equality can be recognized as the curvature (see the equation

$$\kappa(t) = \frac{\dot{x}(t)\ddot{z}(t) - \ddot{x}(t)\dot{z}(t)}{(\dot{x}^2(t) + \dot{z}^2(t))^{3/2}}. \tag{41}$$

in terms of the natural parameter s.

Mylar Balloon and Elastic Curves. Intrinsic Equation of the Elastica

As we know this means to find the equation connecting the curvature κ and the natural parameter s of the curve. For that purpose we will use the Frenet-Serret equations

$$\dot{\mathbf{x}}(s) = \mathbf{T}(s), \qquad \mathbf{N} = \mathbf{T}^{\perp}, \qquad \dot{\mathbf{T}} = \kappa\mathbf{N}, \qquad \dot{\mathbf{N}} = -\kappa\mathbf{T}. \tag{42}$$

By definition $\mathcal{C}$ is an elastic curve when the functional of energy $\int \kappa^2(s)\mathrm{d}s$ has a minimum for a fixed length $L = \int \mathrm{d}s$. Let us consider its infinitesimal deformation defined by the formula

$$\tilde{\mathbf{x}}(s) = \mathbf{x}(s) + \varepsilon(s)\mathbf{N} \tag{43}$$

where $\varepsilon(s)$ is an infinitesimal function of the parameter s. Then

$$\dot{\tilde{\mathbf{x}}} = \dot{\mathbf{x}} + \dot{\varepsilon}\mathbf{N} + \varepsilon\dot{\mathbf{N}} = \mathbf{T} + \dot{\varepsilon}\mathbf{N} - \varepsilon\kappa\mathbf{T} = (1 - \varepsilon\kappa)(\mathbf{T} + \dot{\varepsilon}\mathbf{N}). \tag{44}$$

Mylar Balloon and Elastic Curves. Intrinsic Equation of the Elastica

For the last equation we use the fact that $\varepsilon.\dot{\varepsilon}$ is a negligible quantity. Following the same strategy we have

$$\begin{aligned} \mathrm{d}\tilde{s} &= (\mathrm{d}\tilde{\mathbf{x}}.\mathrm{d}\tilde{\mathbf{x}})^{1/2} = (\dot{\tilde{\mathbf{x}}}.\dot{\tilde{\mathbf{x}}})^{1/2}\mathrm{d}s \\ &= (1-\varepsilon\kappa)[(\mathbf{T}+\dot{\varepsilon}\mathbf{N}).(\mathbf{T}+\dot{\varepsilon}\mathbf{N})]^{1/2}\mathrm{d}s = (1-\varepsilon\kappa)\mathrm{d}s \end{aligned}$$

and therefore

$$\tilde{\mathbf{T}} = \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}\tilde{s}} = \frac{\dot{\tilde{\mathbf{x}}}\mathrm{d}s}{\mathrm{d}\tilde{s}} = \frac{(1-\varepsilon\kappa)(\mathbf{T}+\dot{\varepsilon}\mathbf{N})\mathrm{d}s}{(1-\varepsilon\kappa)\mathrm{d}s} = \mathbf{T}+\dot{\varepsilon}\mathbf{N}.$$

The normal vector $\tilde{\mathbf{N}}$ to the deformed curve $\tilde{\mathcal{C}}$ is defined by the condition that it is orthogonal to $\tilde{\mathbf{T}}$ and that the (infinitesimal) deformation is along the direction of $\mathbf{N}$, i.e.,

$$\tilde{\mathbf{N}} = \mathbf{N} + \lambda\mathbf{T}$$

where λ is function of s.

Mylar Balloon and Elastic Curves. Intrinsic Equation of the Elastica

According to the said above we have

$$\tilde{\mathbf{N}}.\tilde{\mathbf{T}} = (\mathbf{N} + \lambda\mathbf{T})(\mathbf{T} + \dot{\varepsilon}\mathbf{N}) = \lambda + \dot{\varepsilon} = 0$$

and therefore

$$\tilde{\mathbf{N}} = \mathbf{N} - \dot{\varepsilon}\mathbf{T}.$$

Imposing the condition that the deformed curve has a fixed length, i.e.,

$$\int \mathrm{d}\tilde{s} = \int (1 - \varepsilon\kappa)\mathrm{d}s = \int \mathrm{d}s$$

leads to its analytical form presented by the equation

$$\int \varepsilon\kappa \mathrm{d}s = 0.$$

According to the Frenet-Serret formulas (42) another equation has to be satisfied as well, i.e.,

$$\frac{\mathrm{d}\tilde{\mathbf{T}}}{\mathrm{d}\tilde{s}} = \tilde{\kappa}\tilde{\mathbf{N}} = \tilde{\kappa}(\mathbf{N} - \dot{\varepsilon}\mathbf{T}).$$

Mylar Balloon and Elastic Curves. Intrinsic Equation of the Elastica

The left hand side can be rewritten as

$$\frac{\mathrm{d}\tilde{\mathbf{T}}}{\mathrm{d}\tilde{s}} = \frac{(\mathbf{T}+\dot{\varepsilon}\mathbf{N})^{\cdot}\mathrm{d}s}{(1-\varepsilon\kappa)\mathrm{d}s} = (1+\varepsilon\kappa)(\kappa\mathbf{N}+\ddot{\varepsilon}\mathbf{N}-\dot{\varepsilon}\kappa\mathbf{T}) = (\kappa+\ddot{\varepsilon}+\varepsilon\kappa^2)(\mathbf{N}-\dot{\varepsilon}\mathbf{T})$$

which means, that the curvature of the deformed curve is given by the formula

$$\tilde{\kappa} = \kappa + \varepsilon\kappa^2 + \ddot{\varepsilon}.$$

Respectively, as $\tilde{\mathcal{C}}$ is an extremal we have

$$\begin{aligned}
\int \tilde{\kappa}^2\mathrm{d}\tilde{s} + 2\sigma\int \mathrm{d}\tilde{s} &= \int (\kappa+\varepsilon\kappa^2+\ddot{\varepsilon})^2(1-\varepsilon\kappa)\mathrm{d}s + 2\sigma\int(1-\varepsilon\kappa)\mathrm{d}s \\
&= \int \kappa^2\mathrm{d}s + 2\sigma\int \mathrm{d}s + \int(2\kappa\ddot{\varepsilon}+\varepsilon\kappa^3)\mathrm{d}s - 2\sigma\int\varepsilon\kappa\mathrm{d}s \\
&= \int \kappa^2\mathrm{d}s + 2\sigma\int \mathrm{d}s + \int(2\ddot{\kappa}+\kappa^3-2\sigma\kappa)\varepsilon\mathrm{d}s.
\end{aligned}$$

Mylar Balloon and Elastic Curves. Intrinsic Equation of the Elastica

In order to obtain the last formula we have integrated twice by parts $\kappa\ddot{\varepsilon}$. Because $\mathcal{C}$ is an elastic curve the integral $\int \kappa^2 \mathrm{d}s$ has a minimal value for all deformations which preserve its length $\int \mathrm{d}s$ and therefore we have the equation

$$2\ddot{\kappa} + \kappa^3 - 2\sigma\kappa = 0 \tag{45}$$

where σ is a constant. A direct consequence from the above equation is that the curve $\mathcal{C}$ is a critical point for the elastica's functional $\int (k^2 + 2\sigma)\mathrm{d}s$. We are not going to discuss the question which critical points are minimum and which are not, but we will present another derivation of equation (45) later on.
Before that in the next section we will apply the so developed variational techniques to a problem which is quite interesting in both geometrical and mechanical aspects.

Mylar Balloon and Elastic Curves. A Hanging Chain

The potential energy of any infinitesimal element of a homogeneous freely hanging heavy chain is proportional to the length $\mathrm{d}s$ of this element and its height, i.e.,

$$\mathrm{d}U = z\mathrm{d}s.$$

For the entire segment between the points A and B the potential energy is given by the integral

$$J = \int_A^B z\mathrm{d}s$$

and it is assumed that the linear density of the chain mass and the gravitation constant are equal to one.

Mylar Balloon and Elastic Curves.
A Hanging Chain

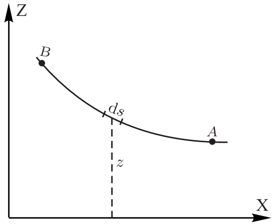

Figure: A homogeneous chain in a gravitational field.

The equilibrium
state of the chain is defined
as the stationary points
of J, i.e., the points for which

$$\delta J = 0 \qquad (46)$$

where δ is the infinitesimal
variation of the functional J.

Mylar Balloon and Elastic Curves. A Hanging Chain

Before finding them we will rewrite the results derived in previous section in an equivalent form, which is more useful in the case under consideration as follows

$$\delta \mathbf{x} = \varepsilon(s)\mathbf{N}(s), \qquad \delta \mathrm{d}s = -\varepsilon\kappa \mathrm{d}s, \qquad \delta \mathbf{T} = \dot{\varepsilon}\mathbf{N}, \qquad \delta \mathbf{N} = -\dot{\varepsilon}\mathbf{T}.$$

Concerning the equation (46) above we have

$$\begin{aligned} \delta J &= \int \delta z \mathrm{d}s + \int z \delta \mathrm{d}s = \int \varepsilon(s) N_z \mathrm{d}s - \int z\varepsilon\kappa \mathrm{d}s \\ &= \int \varepsilon(s)(N_z - z\kappa)\mathrm{d}s = 0. \end{aligned}$$

As $\varepsilon(s)$ is arbitrary function, the last equation is equivalent to

$$N_z - z\kappa = 0.$$

Mylar Balloon and Elastic Curves. A Hanging Chain

According to Fig. ??, $N_z = \cos\psi$ and after dividing by κ and differentiating with respect to s we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\cos\psi}{\kappa} - z\right) = -\frac{\sin\psi}{\kappa}\dot{\psi} - \frac{\dot{\kappa}\cos\psi}{\kappa^2} - \sin\psi = 0.$$

Taking into account that $\dot{\psi} = \kappa$ we can rewrite this equation as

$$-2\tan\psi = \frac{\dot{\kappa}}{\kappa^2} = \frac{1}{2\kappa^2}\frac{\mathrm{d}\kappa^2}{\mathrm{d}\psi} \tag{47}$$

and integrating to obtain the relation

$$\kappa = c\cos^2\psi \tag{48}$$

in which c is the integration constant. Combining (47) and (48) allow us to find the intrinsic equation of the curve in the form

$$\dot{\kappa}^2 + 4\kappa^4 - 4c^2\kappa^2 = 0.$$

Mylar Balloon and Elastic Curves. A Hanging Chain

The solution to this equation is $\kappa = \frac{c}{1+c^2s^2}$ while for $\psi = \int \kappa \mathrm{d}s$ we have

$$\psi = \arctan cs$$

and respectively

$$\sin\psi = \frac{cs}{\sqrt{1+c^2s^2}}, \qquad \cos\psi = \frac{1}{\sqrt{1+c^2s^2}}.$$

Two additional integrations produce the coordinates of the points on the curve

$$x(s) = \frac{\operatorname{arc}\sinh(cs)}{c}, \qquad z(s) = \frac{\sqrt{1+c^2s^2}}{c}.$$

If we eliminate the parameter s from these two equations, we have the standard formula in the textbooks

$$z(x) = \frac{\sqrt{1+\sinh^2(cx)}}{c} = \frac{\cosh(cx)}{c}. \tag{49}$$

The plot of this function, i.e., the curve describing the form of the chain is presented in Fig. 5.

Mylar Balloon and Elastic Curves. A Hanging Chain

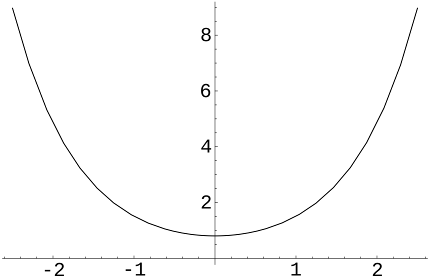

Figure: The graphics of the chain generated via formula (49) for $c = 1.24$.

Mylar Balloon and Elastic Curves. Eulers Elasticas as One-Dimensional Membranes

As it was explained before our problem is to find the extremum of the functional

$$J = \int_0^L \kappa^2(s)\mathrm{d}s + \lambda_1 \int_0^L \cos\theta(s)\mathrm{d}s + \lambda_2 \int_0^L \sin\theta(s)\mathrm{d}s + \lambda \int_0^L \mathrm{d}s \quad (50)$$

which Lagrangian (taking into account that $\kappa(s) = \dfrac{\mathrm{d}\theta(s)}{\mathrm{d}s} = \dot{\theta}(s)$) is

$$F(\theta, \dot{\theta}, s) = \dot{\theta}^2(s) + \lambda_1 \cos\theta(s) + \lambda_2 \sin\theta(s) + \lambda. \quad (51)$$

The respective Euler-Lagrange equation is

$$\ddot{\theta}(s) + \frac{\lambda_1}{2}\sin\theta(s) - \frac{\lambda_2}{2}\cos\theta(s) = 0. \quad (52)$$

Mylar Balloon and Elastic Curves.
Eulers Elasticas as One-Dimensional Membranes

Differentiating with respect to s both sides of the above equation, we have

$$\dddot{\theta}(s) + \frac{\lambda_1}{2}\dot{\theta}(s)\cos\theta(s) + \frac{\lambda_2}{2}\dot{\theta}(s)\sin\theta(s) = 0. \tag{53}$$

On the other side if we multiply both sides of (52) with $\dot{\theta}(s)$ and integrate we have

$$\dot{\theta}^2(s) - \lambda_1\cos\theta(s) - \lambda_2\sin\theta(s) + 2\mu = 0 \tag{54}$$

where 2μ is the integration constant.
The proper combination of equations (53) and (54) leads to the elimination of the Lagrange's multipliers λ_1 and λ_2 and in this way we get the equation

$$\dddot{\theta} + \frac{\dot{\theta}^3}{2} + \sigma\dot{\theta} = 0. \tag{55}$$

Mylar Balloon and Elastic Curves. Eulers Elasticas as One-Dimensional Membranes

Since by definition $\dot{\theta} = \kappa$, we can write aslo

$$\ddot{\kappa} + \frac{\kappa^3}{2} + \mu\kappa = 0 \tag{56}$$

and to see that this is nothing else but the intrinsic equation of the curves we are looking for. Further on we will call it *Euler's elasticas equation*. The case when $\mu \equiv 0$, i.e.,

$$\ddot{\kappa} + \frac{\kappa^3}{2} = 0 \tag{57}$$

is known in the literature as the *free elasticas equation*. A remarcable fact is that solutions of equation (56) can be derived like deformation of the solutions of (57), in other words - the solutions of the free elastcas are suficient to generate the Euler elasticas solutions. J. Bernoulli [1694] claimed this but do not present any arguments. In strictly analitycal form this is proved by Djondjorov *et al* [2009].

Mylar Balloon and Elastic Curves. Eulers Elasticas as One-Dimensional Membranes

It is interesting to mention also that under appropriate choice of the coordinate system the original equation of elasticas (52) can be written in the form

$$\ddot{\theta} + \lambda \sin \theta = 0 \tag{58}$$

which coincides with the equation of the mathematical pendulum. The connection between the pendulunm and elasticas is shown in Fig. 6.

Mylar Balloon and Elastic Curves. Eulers Elasticas as One-Dimensional Membranes

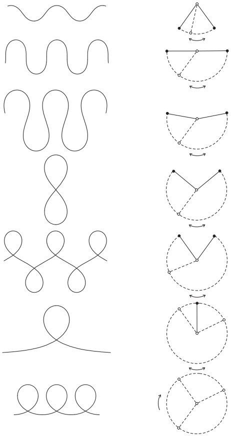

Figure: Examples of conformity between Euler elasticas and the motions of the mathematical pendulum.

Thank you for attention!