The Many Faces of Elastica

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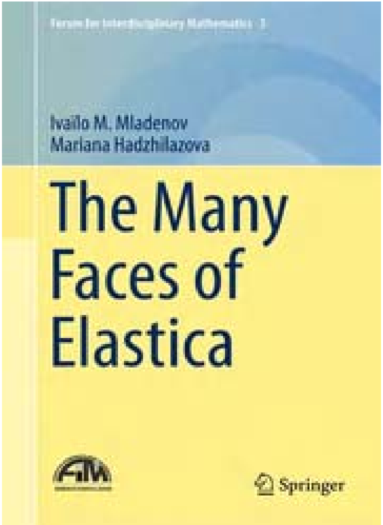
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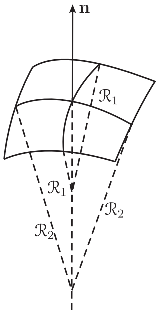
Chapter 5
Equations of Equilibrium States of Membranes

Canham Model



Фигура: A section of a membrane which is curved in two planes. The two curvature centers must lie on the normal to the surface.

The evaluation of the stored elastic energy in a thin plate bended in two perpendicular planes is very complicated. However, if the shear stresses are zero, it would appear that the problem reduces to the integral of the sum of the squares of the two curvatures (Fig. 1). It has been necessary to deviate considerably from the equations of structural engineering in order to accommodate the abundant biological observations. E.g., the red blood cells (RBC) can adopt so many shapes without hemolysis and can change from the crenated form into a biconcave shape.

Canham Model

To solve concretely the problem with RBC shape Canham [1970] had considered the membrane's elastic energy of bending in the form

$$U = \frac{D}{2} \int (\kappa_1^2 + \kappa_2^2) \mathrm{d}\mathcal{A} \tag{1}$$

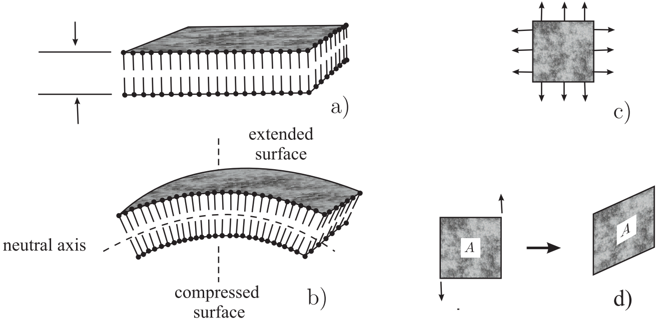
where D is the bending rigidity, and κ_1 and κ_2 are the principle curvatures of the surface based on the model described in the next section.

Key Assumptions in the Model

1) The membrane consists mainly of two isotropic labile surfaces - an interpretation supported by the electron micrographs of membranes and the viscoelastic studies by Rand [1964]. It is acknowledged that micellar structures also exist in the membrane but since they are less stable, Davison & Danielli [1952] assumed that the greater part of the membrane surface has the bimolecular leaflet arrangement.

The interpretation of the viscoelastic nature of the red-cell membrane is illustrated in Fig. 2 (one might consider the stress state in the membrane as a hydrostatic in two dimensions). The flat membrane can resist to distortion only temporarily, without area changes, in the form of a viscous resistance, but can resist to the changes of the area. Any attempt to change the area of either membrane side leads to an elastic resistance to bending because the inner surface is compressed and the outer surface is extended (Fig. 2).

Key Assumptions in the Model



Фигура: A schematic interpretation of the RBC membrane. a) leaflet model proposed by Davison & Danielli. b) An element of membrane has been bent in one plane, showing the compression of the inner surface and the stretching of the outer surface. c) An element of the area which is stressed equally in all directions in the plane of the membrane (a two-dimensional hydrostatic stress) will resists to the stress elastically. d) The membrane will offer only viscous resistance to shear stresses, resulting in deformation with no storage of elastic energy.

Key Assumptions in the Model

2) The red cell is not assumed to be cast in the biconcave shape but rather that the biconcave shape minimizes the elastic energy stored in the membrane. We assumed that an element of the membrane has no stored bending energy if it is flat, and that any curved element of membrane has stored elastic energy.
3) Any change of the shape due to the osmotic volume shifts are considered to take place without altering the total cellular area (that is the area of the surface through the neutral axis (Fig. 2)).
4) The membrane has the same physical properties over the entire surface. This concurs with the results of Rand & Burton [1964], but contradicts those of Murphy [1965] and Fung & Tong [1968].

Helfrich and Deuling Model

To describe the shape of RBC Helfrich & Deuling [1975] suggest a generalization of the Cahnam's model.
In this model the energy density per unit area has the form

$$\mathcal{E} = \frac{1}{2}k_c(\kappa_1 + \kappa_2 - \mathbb{h})^2 + \frac{1}{2}\bar{k}_c\kappa_1\kappa_2 \tag{2}$$

where κ_1 and κ_2 are the principal curvatures, k_c and $\bar{k}_c$ are the membrane's elastic moduli and $\mathbb{h}$ is the so called spontaneous curvature.

Helfrich and Deuling Model

For closed surfaces (like RBC) the integral of the second term will be a constant, and consequently, the shape of the membrane is determined only by the first term. Taking into account the axial symmetry of RBC we can write

$$\delta\left(\frac{1}{2}k_c\int(\kappa_\mu+\kappa_\pi-\mathbb{h})^2\mathrm{d}\mathcal{A}+\Delta pV+\lambda\mathcal{A}\right)=0. \qquad (3)$$

Here κ_μ and κ_π are the curvatures along the meridians, respectively the parallels (see (??)), Δp and λ are the Lagrange multipliers.

Helfrich and Deuling Model

If we choose x (the distance from the axis of symmetry OZ) as a parameter, the formulas for the curvatures are

$$\kappa_\mu = \frac{\ddot{z}(x)}{(1+\dot{z}^2(x))^{3/2}}, \qquad \kappa_\pi = \frac{\dot{z}(x)}{x\sqrt{1+\dot{z}^2(x)}}. \tag{4}$$

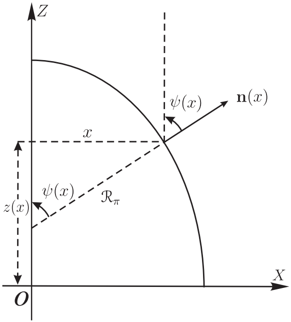
Besides, we have the identity which can be checked directly, i.e.,

$$\kappa_\mu = \frac{\mathrm{d}}{\mathrm{d}x}(x\kappa_\pi) \tag{5}$$

and which expanded form is

$$\frac{\mathrm{d}\kappa_\pi}{\mathrm{d}x} = \frac{\kappa_\mu - \kappa_\pi}{x}. \tag{6}$$

Helfrich and Deuling Model



Фигура: Profile curve and geometry of axially symmetric surface.

If we denote by $\psi(x)$ the angle made by the surface normal and the axis of rotation and consider the triangle in which the hypotenuse is the curvature radius $\mathcal{R}_\pi$ we can write the equation (cf. Fig. 3)

$$\frac{x}{\mathcal{R}_\pi} = \sin\psi \qquad (7)$$

this relation can be immediately transformed in the form

$$\kappa_\pi = \frac{\sin\psi}{x}. \qquad (8)$$

Helfrich and Deuling Model

Combining this result with (5) we have the expression for the other curvature as well, i.e.,

$$\kappa_\mu = \cos\psi \frac{\mathrm{d}\psi}{\mathrm{d}x}. \tag{9}$$

These equations should be supplemented with the geometrical relation

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\tan\psi(x) \tag{10}$$

in which the sigh accounts the fact that when x is increasing z is decreasing.

Helfrich and Deuling Model

Respectively, the volume V and the surface area $\mathcal{A}$ are defined by the integrals

$$V = \int \mathrm{d}V \qquad \text{and} \qquad \mathcal{A} = \int \mathrm{d}\mathcal{A} \tag{11}$$

in which the formulas for the infinitesimal quantities of the volume and the area are

$$\mathrm{d}V = \pi x^2 |\mathrm{d}z| = \frac{\pi x^3 \kappa_\pi \mathrm{d}x}{\sqrt{1 - x^2\kappa_\pi^2}}, \ \mathrm{d}\mathcal{A} = 2\pi x \mathrm{d}s = \frac{2\pi x \mathrm{d}x}{\sqrt{1 - x^2\kappa_\pi^2}} \cdot \tag{12}$$

Helfrich and Deuling Model

Using the relation (6) we can eliminate κ_μ in equation (3) and in this way to transform it into the form

$$\delta \int \frac{x((x\dot{\kappa}_\pi + 2\kappa_\pi - \mathrm{I\!h})^2 + \kappa_\pi + \Delta p x^2 \kappa_\pi / k_c + 2\lambda / k_c)}{\sqrt{1 - x^2\kappa_\pi^2}} \mathrm{d}x = 0 \tag{13}$$

where $\dot{\kappa}_\pi = \dfrac{\mathrm{d}\kappa_\pi}{\mathrm{d}x}$.

The Euler-Lagrange equation corresponding to the integrand in (13) which is already in the canonical form $F = F(x, \kappa_\pi, \dot{\kappa}_\pi)$ can be written as

$$\dot{\kappa}_\mu = \frac{\mathrm{d}\kappa_\mu}{\mathrm{d}x} = \frac{x((\kappa_\pi(\kappa_\pi - \mathrm{I\!h})^2 - \kappa_\mu^2)/2 + (\lambda/k_c)\kappa_\pi + (1/2)\Delta p/k_c)}{\sqrt{1 - x^2\kappa_\pi^2}} - \frac{\kappa_\mu - \kappa_\pi}{x}. \tag{14}$$

Helfrich and Deuling Model

This equation is of the first order because the second derivative of κ_π is expressed via $\dot{\kappa}_\mu$ by making use (twice) of the fundamental geometrical relation (6). Taken together, the equations (6) and (14) form a close system of ODE which can be integrated (numerically). One can find the membrane contour after one more integration, namely

$$z(x) = -\int \tan\psi(x)\mathrm{d}x = -\int \frac{x\kappa_\pi}{\sqrt{1 - x^2\kappa_\pi^2}}\mathrm{d}x. \qquad (15)$$

This model was not developed further on because it is purely numerical and after a few years it was replaced by the Ou-Yang and Helfrich model, which will be considered in the next section.

Ou-Yang and Helfrich Model

The equilibrium shapes of a vesicle in this model (Helfrich [1973]) are determined by the minimization of the shape energy which may be written as

$$F = \frac{1}{2} k_c \oint (\kappa_1 + \kappa_2 - \mathbb{h})^2 \mathrm{d}A + \Delta p \int \mathrm{d}V + \lambda \oint \mathrm{d}A. \qquad (16)$$

Here (as before) $\mathrm{d}A$ and $\mathrm{d}V$ are the infinitesimal elements surface of the area, respectively the volume, k_c the bending rigidity, κ_1 and κ_2 the two principal curvatures, and $\mathbb{h}$ is the spontaneous curvature.

Ou-Yang and Helfrich Model

The last serves to describe the effect of the asymmetry of the membrane or its environment Deuling & Helfrich [1976]. The first term in (16) is the elastic energy of the vesicle. The second and the third terms take into account the constrains of constancy of the volume, respectively the area or represent the actual work of the deformation. Depending on the situation the pressure difference $\Delta p = p_{\rm out} - p_{\rm in}$ and the tensile stress λ serve as Lagrange multipliers or they are prescribed experimentally by measuring the volume or the area. Instead of the last term of equation (16), Jenkins [1977, 1977a] introduces a local area constrain by $\gamma \oint \mathrm{d}A$, where γ is a Lagrangian multiplier which varies with the position. He had derived a general equilibrium equation, but do not consider the spontaneous curvature besides in the special case of the fluctuating sphere (Schneider *et al* [1984]). A generalized equilibrium shape equation including the spontaneous curvature will be derived below.

Ou-Yang and Helfrich Model. Basic Formulas and Definitions

Theoretically, the membrane of any vesicle may be represented as a closed surface in the Euclidean three space given by the vector $\mathbf{x}(u, v)$ depending on the two real parameters u, v. We introduce the following quantities which will be used further (see **??**)

$$
\begin{array}{lll}
\mathbf{x}_i = \partial_i \mathbf{x}, & \mathbf{x}_{ij} = \partial_i \partial_j \mathbf{x}, & g_{ij} = \mathbf{x}_i . \mathbf{x}_j \\
g^{ij} = (g_{ij})^{-1}, & g = \det(g_{ij}), & h_{ij} = \mathbf{x}_{ij} . \mathbf{n} \\
h^{ij} = (h_{ij})^{-1}, & h = \det(h_{ij}), & i, j = u, v = 1, 2
\end{array}
\tag{17}
$$

where $\partial_1 = \partial_u$, $\partial_2 = \partial_v$, and the matrices g_{ij} and h_{ij} are associated with the first and the second fundamental forms of the surface.

Ou-Yang and Helfrich Model. Basic Formulas and Definitions

The outward unit normal vector $\mathbf{n}$, and the Christoffel symbols Γ^k_{ij} are defined by (cf. (??))

$$\mathbf{n} = (\mathbf{x}_1 \times \mathbf{x}_2)/\sqrt{g}, \qquad \mathbf{x}_{ij} = \Gamma^k_{ij}\mathbf{x}_k + h_{ij}\mathbf{n}. \tag{18}$$

Here and in the following the repeated indices imply summation over them. The mean and Gaussian curvatures can be written respectively as

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}g^{ij}h_{ij}, \qquad K = \kappa_1\kappa_2 = h/g. \tag{19}$$

Ou-Yang and Helfrich Model. Basic Formulas and Definitions

We assume $\mathbf{x}$ to be an equilibrium shape and consider a slightly distorted surface defined by the equations

$$\tilde{\mathbf{x}} = \mathbf{x} + \psi(u, v)\mathbf{n} \tag{20}$$

where $\psi(u, v)$ is sufficiently smooth function. Using equations (17)-(20), we may calculate $\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_{ij}, \tilde{g}_{ij}, \tilde{h}_{ij}$ and so on. For example we have

$$\tilde{\mathbf{x}}_i = \mathbf{x}_i + \psi_i \mathbf{n} + \psi \partial_i \mathbf{n} \tag{21}$$

where $\psi_i = \partial_i \psi$. By using the Weingarten equations (cf. (??))

$$\partial_i \mathbf{n} = -h_{ij} g^{jk} \mathbf{x}_k \tag{22}$$

we can transform the expression at the most right hand side of (21) into

$$\tilde{\mathbf{x}}_i = \mathbf{x}_i + \psi_i \mathbf{n} - \psi h_{ij} g^{jk} \mathbf{x}_k. \tag{23}$$

Ou-Yang and Helfrich Model. Basic Formulas and Definitions

Further on, by using the identities

$$h_{ij}g^{jk}h_{kl} = 2Hh_{il} - Kg_{il} \tag{24}$$

and $\mathbf{x}_i.\mathbf{n} = 0$ allow us to write

$$\delta g_{ij} = \tilde{\mathbf{x}}_i.\tilde{\mathbf{x}}_j - \mathbf{x}_i.\mathbf{x}_j = -2\psi h_{ij} + \psi_i\psi_j + \psi^2(2Hh_{ij} - Kg_{ij}). \tag{25}$$

The identity

$$g = \det(g_{ij}) = \frac{1}{2}\varepsilon_{3ij}\varepsilon_{3kl}g_{ik}g_{jl} \tag{26}$$

in combination with (25) results in

$$\delta g = g(-4\psi H + g^{ij}\psi_i\psi_j + \psi^2(4H^2 + 2K)) + \mathcal{O}(\psi^3). \tag{27}$$

Ou-Yang and Helfrich Model. Basic Formulas and Definitions

Here and in the following $\mathcal{O}(\psi^3)$ refers to terms of higher than quadratic order in ψ and the symbols e_{ijk} are defined as

$$e_{ijk} = \begin{cases} +1, & \text{when } ijk \text{ is an even permutation of } 123 \\ -1, & \text{when } ijk \text{ is an odd permutation of } 123 \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

In addition, we have the variations

$$\begin{aligned} \delta g^{ij} &= 2\psi(2Hg^{ij} - Kh^{ij}) + \left(\frac{1}{g}\varepsilon_{3ik}\varepsilon_{3jl} - g^{ij}g^{kl}\right)\psi_k\psi_l \\ &\quad -3\psi^2\left((K-4H^2)g^{ij} + 2HKh^{ij}\right) + \mathcal{O}(\psi^3) \end{aligned} \tag{29}$$

$$\begin{aligned} \delta h_{ij} &= \psi_{ij} + \psi(Kg_{ij} - 2Hh_{ij}) - \Gamma^k_{ij}\psi_k \\ &\quad + \psi\psi_m\left(K\Gamma^k_{ij}\varepsilon_{3lk}\varepsilon_{3mp}g_{pq}h^{ql} + (h_{ij}g^{lm})_l + h_{jk}g^{kl}\Gamma^m_{li}\right) \\ &\quad + \psi_k\psi_m\left(g^{lk}(\delta_{im}h_{jl} + \delta_{jm}h_{il} - \frac{1}{2}h_{ij}g^{mk}\right) + \mathcal{O}(\psi^3). \end{aligned} \tag{30}$$

Ou-Yang and Helfrich Model. Basic Formulas and Definitions

Taking into account the equations $(h_{jl}g^{lm})_i = \partial_i(h_{jl}g^{lm})$, the definition of the Kronecker symbol

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

and equations (19), (29) and (30) one can obtain the variation of the mean curvature

$$\begin{aligned}\delta H &= \psi(2H^2 - K) + \frac{1}{2}g^{ij}\nabla_i\psi_j + \psi^2(4H^3 - 3HK) \\ &\quad + \frac{1}{2}\psi\psi_m\left(g^{ij}(h_{jl}g^{lm})_i - h_{lk}g^{km}g_{ij}\Gamma^h_{ij} + (Kh^{ij} - 2Hg^{ij})\Gamma^m_{ij}\right) \\ &\quad + \frac{1}{2}\psi_i\psi_j(Hg^{ij} - Kh^{ij}) + \psi\psi_{ij}(2Hg^{ij} - Kh^{ij}) + \mathcal{O}(\psi^3)\end{aligned} \quad (31)$$

where $\nabla_i\psi_j$ is the covariant derivative of ψ_j defined by

$$\nabla_i\psi_j = \psi_{ij} - \Gamma^k_{ij}\psi_k. \quad (32)$$

Ou-Yang and Helfrich Model. Basic Formulas and Definitions

The local variation of the surface area is given by

$$\delta\sqrt{g} = (-2\psi H + \frac{1}{2}g^{ij}\psi_i\psi_j + \psi^2 K)\sqrt{g} + \mathcal{O}(\psi^3) \qquad (33)$$

and the global one by the formula

$$\delta\mathcal{A} = \delta\oint \mathrm{d}\mathcal{A} = \oint(-2\psi H + \frac{1}{2}g^{ij}\psi_i\psi_j + \psi^2 K)\mathrm{d}\mathcal{A} + \mathcal{O}(\psi^3). \quad (34)$$

The variation of the volume is found to be

$$\delta V = \oint(\psi - \psi^2 H)\mathrm{d}\mathcal{A} + \mathcal{O}(\psi^3) \qquad (35)$$

and evidently all variations can be expressed via H, K, g_{ij}, h_{ij} and Γ_{ij} that have been just derived.

Ou-Yang and Helfrich Model. Shape Equation

In order to obtain the equation of mechanical equlibrium of the vesical membrane we have to calculate the first variation of the shape energy given by equation (16). Because of (19) we may write

$$\delta F = \frac{1}{2} k_c \delta \oint (2H + \mathrm{lh})^2 \mathrm{d}\mathcal{A} + \Delta p \, \delta \int \mathrm{d}V + \lambda \delta \oint \mathrm{d}\mathcal{A}. \tag{36}$$

The first variations of V and $\mathcal{A}$ are defined in (34) and (35) and respectively

$$\delta \oint \mathrm{d}\mathcal{A} = -\oint 2\psi H \mathrm{d}\mathcal{A} \tag{37}$$

and

$$\delta \int \mathrm{d}V = \oint \psi \mathrm{d}\mathcal{A}. \tag{38}$$

Ou-Yang and Helfrich Model. Shape Equation

The first variation of the curvature-elastic energy may be written as

$$\begin{aligned} \delta F_c &= \frac{1}{2} k_c \delta \oint (2H + \mathrm{Ih})^2 \mathrm{d}\mathcal{A} \\ &= \frac{1}{2} k_c \oint \left((2H + \mathrm{Ih})^2 \delta \mathrm{d}\mathcal{A} + 4(2H + \mathrm{Ih}) \delta H \, \mathrm{d}\mathcal{A}\right) . \end{aligned} \tag{39}$$

From (31) we have

$$\delta H = \psi(2H^2 - K) + \frac{1}{2} g^{ij} \nabla_i \psi_j \tag{40}$$

which with (32) becomes

$$\delta H = \psi(2H^2 - K) + \frac{1}{2} g^{ij} (\psi_{ij} - \Gamma_{ij}^k \psi_k). \tag{41}$$

Ou-Yang and Helfrich Model. Shape Equation

Inserting (37) and (41) into (39), integrating ψ_{ij} and ψ_k by parts, and using these results, equations (38) and (37) we obtain

$$\begin{aligned} \delta F \quad = \quad & \oint \psi \left(\Delta p - 2\lambda H + k_c(2H + \mathbb{h})(2H^2 - 2K - \mathbb{h}H) \right. \\ & \left. +(k_c/\sqrt{g})(\partial_i\partial_j + \partial_k\Gamma^k_{ij})g^{ij}(2H + \mathbb{h})\sqrt{g} \right) \mathrm{d}A. \end{aligned} \tag{42}$$

Additionally, by using equations (17) and (18) one may prove the identity

$$\partial_i \left((\partial_i g^{ij}\sqrt{g})f \right) = -\partial_k(\Gamma^k_{ij}g^{ij}\sqrt{g}f) \tag{43}$$

in which $f(u, v)$ is an arbitrary function.

Ou-Yang and Helfrich Model. Shape Equation

Transforming the last term in equation (42) appropriately, we have

$$\delta F = \oint \psi \left(\Delta p - 2\lambda H + k_c(2H + \mathrm{h})(2H^2 - 2K - \mathrm{h}H) + 2k_c \Delta_{\mathcal{S}} H\right) \mathrm{d}A \qquad (44)$$

where $\Delta_{\mathcal{S}}$ (in front of H) is the Laplace-Beltrami operator of the surface $\mathcal{S}$, i.e.,

$$\Delta_{\mathcal{S}} = (1/\sqrt{g})\partial_i(g^{ij}\sqrt{g}\partial_j). \qquad (45)$$

Since $\mathbf{x}(u, v)$ describes an equilibrium shape and if $\delta F = 0$ is satisfied for any infinitesimal function $\psi(u, v)$, equation (44) reduces to the equilibrium condition (Ou-Yang & Helfrich [1989])

$$2k_c \Delta_{\mathcal{S}} H - 2\lambda H + k_c(2H^2 - 2K - \mathrm{h}H)(2H + \mathrm{h}) + \Delta p = 0. \qquad (46)$$

Symmetries of the Shape Equation. Cartesian Coordinates

Now we will consider the symmetries of the shape equation (46). Using the standard approach (see Ovsiannikov [1982], Ibragimov [1985], Bluman & Kumei [1989] and Olver [1993]) for group analysis of differential equations we can proof that in Monge parameterization (see Example **??**) the symmetries of (46) coincide with the full group of motions in $\mathbb{R}^3$. The generators and their characteristics are represented in Table 1. It was proved (see Vassilev *et al* [2006]), that all symmetries of equation (46) are variational as well, i.e., they are symmetries of the functional

$$\mathcal{F} = \mathcal{F}_c + \lambda \int_{\mathcal{S}} \mathrm{d}\mathcal{A} + p \int \mathrm{d}V \tag{47}$$

in which λ and p are real constants, that denote the surface tension and the osmotic pressure difference on both sides (inside and outside) of the membrane, and $\mathrm{d}V$ is the volume element.

Symmetries of the Shape Equation. Cartesian Coordinates

Таблица: Generators and characteristics of the group of motions in $\mathbb{R}^3$.

Generators	Characteristics
Translations	
$\mathbf{v}_1 = \frac{\partial}{\partial x^1}$	$Q_1 = -w_1$
$\mathbf{v}_2 = \frac{\partial}{\partial x^2}$	$Q_2 = -w_2$
$\mathbf{v}_3 = \frac{\partial}{\partial w}$	$Q_3 = 1$
Rotations	
$\mathbf{v}_4 = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$	$Q_4 = x^2 w_1 - x^1 w_2$
$\mathbf{v}_5 = -w \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial w}$	$Q_5 = x^1 + w w_1$
$\mathbf{v}_6 = -w \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial w}$	$Q_6 = x^2 + w w_2$

Symmetries of the Shape Equation. Cartesian Coordinates

Respectively, in accordance with the Noether's theorem there are six linearly independent conservation laws

$$D_\alpha P_j^\alpha = Q_j E(L), \qquad \alpha = 1, 2, \qquad j = 1, \ldots, 6$$

where

$$P_j^\alpha = N_j^\alpha L$$

$$\begin{aligned}
N_j^\alpha &= \xi_j^\alpha + Q_j \frac{\partial}{\partial w_\alpha} - \frac{1}{2} Q_j D_\mu \frac{\partial}{\partial w_{\alpha\mu}} - \frac{1}{2} Q_j D_\mu \frac{\partial}{\partial w_{\mu\alpha}} \\
&\quad + \frac{1}{2} \left(D_\mu Q_j\right) \frac{\partial}{\partial w_{\alpha\mu}} + \frac{1}{2} \left(D_\mu Q_j\right) \frac{\partial}{\partial w_{\mu\alpha}} \\
D_\alpha &= \frac{\partial}{\partial x^\alpha} + w_\alpha \frac{\partial}{\partial w} + w_{\alpha\mu} \frac{\partial}{\partial w_\mu} + \cdots \\
E &= \frac{\partial}{\partial w} - D_\mu \frac{\partial}{\partial w_\mu} + D_\mu D_\nu \frac{\partial}{\partial w_{\mu\nu}} - \cdots .
\end{aligned}$$

Symmetries of the Shape Equation. Group-Invariant Solutions

It is easy to check that the vector fields

$$\langle \mathbf{v}_1 \rangle \,, \qquad \langle a\mathbf{v}_3 + \mathbf{v}_4 \rangle$$

where a is a real constant form an optimal system of one dimensional subalgebras of the algebra of the symmetry group of the shape equation. Therefore the different group-invariant solutions to equation (46) correspond to the vector fields $\mathbf{v}_1$ and $a\mathbf{v}_3 + \mathbf{v}_4$ and we can obtain every group-invariant solution by them. The two types of the reduced group-invariant equations which determine the above group-invariant solutions are

Symmetries of the Shape Equation. Group-Invariant Solutions

1) $G(\mathbf{v}_1)$ – invariant solution of the type

$$w = W\left(x^1\right)$$

of the reduced equation

$$\frac{k_c}{\left(v^2+1\right)^{5/2}} v_{11} - \frac{5 k_c v}{2\left(v^2+1\right)^{7/2}} v_1^2 - \frac{1}{2}\frac{\left(k_c \mathrm{lh}^2 + 2\lambda\right) v}{\left(v^2+1\right)^{1/2}} + p\, x^1 = C_1 \tag{48}$$

in which C_1 is a real number, $v = \mathrm{d}W/\mathrm{d}x^1$, $v_1 = \mathrm{d}v/\mathrm{d}x^1$ and $v_{11} = \mathrm{d}v_1/\mathrm{d}x^1$.

Symmetries of the Shape Equation. Group-Invariant Solutions

2) $G\,(a\mathbf{v}_3 + \mathbf{v}_4)$–invariant solution of the type

$$w = \widehat{w}\,(r) + a\arctan(\frac{x^2}{x^1}), \qquad r = \sqrt{(x^1)^2 + (x^2)^2}$$

of the reduced equation written in the form

$$E_r(L_2) + p\,r = C_2$$

where

$$E_r = \frac{\partial}{\partial\, v} - \left(\frac{\partial}{\partial\, r} + v_1\frac{\partial}{\partial v} + v_{11}\frac{\partial}{\partial\, v_1}\right)\frac{\partial}{\partial\, v_1}$$

$$L_2 = \frac{k_c F^2}{G^5} + \frac{2k_c \mathbb{h} F}{G^2} + \left(k_c \mathbb{h}^2 + 2\lambda\right) G$$

$$G^2 = r^2\left(v^2+1\right) + a^2, \qquad F = r\left(r^2 + a^2\right) v_1 + \left(G^2 + a^2\right) v$$

and additionally one should has in mind that C_2 is a real number, $v = \mathrm{d}\widehat{w}/\mathrm{d}r$, and $v_1 = \mathrm{d}v/\mathrm{d}r$.

Symmetries of the Shape Equation. Conformal Coordinates

The shape equation in Monge representation is a fourth order nonlinear partial differential equation. If we turn to a conformal metric on the plane and change appropriately the variables the derivatives in the equation decrease their order. Simultaneously we have to add three more equations – the so called compatibility conditions of G-C-M (see equations (**??**)). In general terms, this is the price that we have to pay for lowering the order of the equation. In this setting we face the problem of finding the solutions of a system of four second order partial differential equations about four functions of two independent variables (for additional information see de Matteis [2003] and Pulov *et al* [2012]).

Symmetries of the Shape Equation. Conformal Coordinates

It turns out that it is convenient to choose the conformal coordinates in which the metric and matrix h of the second fundamental form $h_{ij}\mathrm{d}x^i\mathrm{d}x^j$ (the repeated indices means that we have summation) take the form

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = 4q^2\varphi^2(\mathrm{d}x^2+\mathrm{d}z^2), \qquad i,j = 1,2, \quad x^1 = x,\ x^2 = z$$

$$h = \begin{pmatrix} \theta & \omega \\ \omega & 8q^2\varphi(1+\mathbb{h}\varphi)-\theta \end{pmatrix}.$$

Here q, φ, θ and ω are smooth functions of conformal coordinates x and z, and $\mathbb{h}$ is the spontaneous curvature which is connected with the mean curvature H by the formula

$$H = \frac{1}{\varphi} + \mathbb{h}. \tag{49}$$

Symmetries of the Shape Equation. Conformal Coordinates

Applying the Brioschi's formula for the Gausian curvature

$$K = -\Delta \log(2q\varphi) \tag{50}$$

in which the explicit form (cf. formula (45)) of the Laplace-Beltrami operator Δ is

$$\Delta = \frac{1}{4q^2\varphi^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \tag{51}$$

we have

$$K = \frac{1}{4q^4\varphi^2}(q_x^2 + q_z^2) + \frac{1}{4q^2\varphi^4}(\varphi_x^2 + \varphi_z^2) - \frac{\Delta q}{q} - \frac{\Delta\varphi}{\varphi}.$$

Symmetries of the Shape Equation. Conformal Coordinates

The change of the variables in the shape equation results in a system of four partial differential equations

$$
\begin{aligned}
q^2\Delta\varphi + 2q\varphi\Delta q - \frac{1}{2q^2\varphi}(q_x^2+q_z^2) + \frac{q^2}{4\varphi^2}(8\varphi+\alpha_2\varphi^2+\alpha_3\varphi^3+\alpha_4\varphi^4) &= 0\\
\theta_z-\omega_x-(8+\frac{\alpha_2}{3}\varphi)q(\varphi q_z+q\varphi_z) &= 0\\
\omega_z+\theta_x-\frac{\alpha_2}{3}q\varphi(\varphi q_x+q\varphi_x)-8q\varphi q_x &= 0\\
4q^2\varphi\Delta\varphi+4q\varphi^2\Delta q-\frac{q_x^2+q_z^2}{q^2}-\frac{\varphi_x^2+\varphi_z^2}{\varphi^2}+(2+\frac{\alpha_2}{12}\varphi)\frac{\theta}{\varphi}-\frac{\omega^2+\theta^2}{4q^2\varphi^2} &= 0
\end{aligned}
\tag{52}
$$

in which

$$
\alpha_2 = 24\mathbb{h},\qquad \alpha_3 = 8(2\mathbb{h}^2-\frac{\lambda}{k}),\qquad \alpha_4=\frac{4p}{k}-\frac{8\lambda\mathbb{h}}{k}.
$$

Symmetries of the Shape Equation. Lie Equations

The principle aim of the group analysis of any system of differential equations is to find the maximal group of continuous transformations (the Lie group of point transformation) of dependent and independent variables for which this system is invariant. This is the so called symmetry group (admissible group) of the system of differential equations. Since every Lie group is generated by its one-parameter subgroups, the problem is to find all groups of one-parameter transformations which keep the system unchanged.

Symmetries of the Shape Equation. Lie Equations

The one-parameter groups are defined by the system of the so called Lie equations

$$\frac{\mathrm{d}\Phi^i}{\mathrm{d}\varepsilon} = \xi^i(\Phi,\Psi), \quad \tilde{x}^i = \Phi^i(x,u,\varepsilon), \quad \Phi^i|_{\varepsilon=0} = x^i, \quad i = 1,2$$
$$\frac{\mathrm{d}\Psi^\alpha}{\mathrm{d}\varepsilon} = \eta^\alpha(\Phi,\Psi), \quad \tilde{u}^\alpha = \Psi^\alpha(x,u,\varepsilon), \quad \Psi^\alpha|_{\varepsilon=0} = u^\alpha, \quad \alpha = 1,2,3,4 \tag{53}$$

generating the flow of the vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{54}$$

in the space of independent and dependent variables (x,u). Here ε is the parameter of the group ($\varepsilon \in I \subset \mathbb{R}$, $0 \in I$), and x, u, Φ and Ψ are respectively vectors with coordinates x^i, u^α, Φ^i and Ψ^α.

Symmetries of the Shape Equation. Lie Equations

The vector X is the so called infinitesimal operator or generator of the symmetry group. Its second prolongation

$$\mathrm{pr}^{(2)}X \equiv \tilde{X} = X + \eta_\alpha^x \frac{\partial}{\partial u_x^\alpha} + \eta_\alpha^z \frac{\partial}{\partial u_z^\alpha} + \eta_\alpha^{xx} \frac{\partial}{\partial u_{xx}^\alpha} + \eta_\alpha^{xz} \frac{\partial}{\partial u_{xz}^\alpha} + \eta_\alpha^{zz} \frac{\partial}{\partial u_{zz}^\alpha}$$

is the necessary ingredient for applying of the Lie group techniques for analysis of the systems of differential equations. The coefficients η_α^x, $\eta_\alpha^{xx}, \ldots$ are expressed by the first and the second order derivatives of $\xi^i(x,u)$, $u^\alpha(x,u)$ and $\eta^\alpha(x,u)$.

Symmetries of the Shape Equation. Determining System of Equation

At the first stage of group analysis we create and solve the so called determining system of equations

$$\tilde{X}[F_\nu] = 0, \qquad F_\nu = 0, \qquad \nu = 1, \ldots, 4 \tag{55}$$

where F_ν, $\nu = 1, \ldots, 4$, denote the expressions in the left hand side of the system of equation (52). The huge number of equations in determining system is an origin of serious technical difficulties. These difficulties can be at least partially surmounted by using some of the contemporary computer algebra systems. For the system (52) we have used the program *LieSymm-PDE* (see Pulov *et al* [2007]). In this way we have obtained determining system consisting of 206 partial differential equations.

Symmetries of the Shape Equation. Determining System of Equation

After 17 contiguous executes of the program in interactive mode the determining system was solved. The additional program modules were created, and each of them solves a special kind of algebraic or differential equations. After each restart of the program one introduces additional data – and these are either some algebraic or differential consequences of the equations that have still to be solved or some solutions of them. At the end of each stage of the program the functions ξ^i and η^α have new form which approaches the sought solution – the generator of the symmetry group (52). The number of equations in determining system decrease – after the third execution of the program this number is 46, then it stays at 30 and after that quickly reduces to zero (see Pulov *et al* [2012]).

Symmetries of the Shape Equation. Determining System of Equation

At the end for infinitesimal operator $\bar{X}$ of the system of point symmetry group (52) we have the expression

$$\bar{X} = X_1 - q\xi^1_x \frac{\partial}{\partial q} - 2(\theta\xi^1_x + \omega\xi^2_x)\frac{\partial}{\partial\theta} - 2\left(\omega\xi^1_x - \left(\theta - 4q^2\varphi - \frac{\alpha_2}{6}q^2\varphi^2\right)\xi^2_x\right)\frac{\partial}{\partial\omega}$$

where $X_1 = \xi^1\dfrac{\partial}{\partial x} + \xi^2\dfrac{\partial}{\partial z}$, and $\xi^1 = \xi^1(x,z)$ Pч $\xi^2 = \xi^2(x,z)$ are arbitrary functions, satisfying the Cauchy–Riemann condition

$$\frac{\partial\xi^1}{\partial z} = -\frac{\partial\xi^2}{\partial x}, \qquad \frac{\partial\xi^1}{\partial x} = \frac{\partial\xi^2}{\partial z}. \tag{56}$$

Thank you for attention!