Geometry and Exact Soliton Solutions of the Integrable $su(3)$-Spin Systems

Akbota Myrzakul

Eurasian national university, Astana, Kazakhstan

Varna, Bulgaria 2018

Introduction

The vector nonlinear Schrödinger equation is associated with symmetric space $SU(n+1)/S(U(1)\otimes U(n))$. The special case $n=2$ of such symmetric space is associated with the Manakov system

$$
\begin{aligned}
iq_{1t}+q_{1xx}+2(|q_1|^2+|q_2|^2)q_1 &= 0, &(1)\\
iq_{2t}+q_{2xx}+2(|q_1|^2+|q_2|^2)q_2 &= 0. &(2)
\end{aligned}
$$

Geometrical equivalent counterpart of the Manakov system is the coupled spin systems

$$
\begin{aligned}
\mathbf{A}_t+\mathbf{A}\wedge\mathbf{A}_{xx}+u_1\mathbf{A}_x+2v_1\mathbf{H}\wedge\mathbf{A} &= 0, &(3)\\
\mathbf{B}_t+\mathbf{B}\wedge\mathbf{B}_{xx}+u_2\mathbf{B}_x+2v_2\mathbf{H}\wedge\mathbf{B} &= 0, &(4)
\end{aligned}
$$

where spin vectors $\mathbf{A}=(A_1,A_2,A_3)$ and $\mathbf{B}=(B_1,B_2,B_3)$, $\mathbf{A}^2=\mathbf{B}^2=1$, $\mathbf{H}=(0,0,1)^T$ is the constant magnetic field, u_j and v_j are coupling potentials.

In matrix form the coupled spin systems (3)-(4)

$$iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + v_1[\sigma_3, A] = 0, \quad (5)$$

$$iB_t + \frac{1}{2}[B, B_{xx}] + iu_2 B_x + v_2[\sigma_3, B] = 0, \quad (6)$$

where

$$A = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \quad A^2 = I = diag(1,1), \quad A^\pm = A_1 \pm iA_2. \quad (7)$$

$$B = \begin{pmatrix} B_3 & B^- \\ B^+ & -B_3 \end{pmatrix}, B^2 = I = diag(1,1), \quad B^\pm = B_1 \pm iB_2. \quad (8)$$

Lax representation of the coupled spin equation hierarchy

We present two possible versions of the Lax representation (LR) for the coupled spin equation hierarchy.

LR type - I

The first type of LR for the coupled spin equation hierarchy reads as

$$Y_x = -i\lambda P Y, \quad (9)$$

$$Y_t = \sum_{j=1}^{N} \lambda^j V_j Y, \quad (10)$$

where λ is a spectral parameter and

$$P = \frac{1}{2+K}\begin{pmatrix} 2A_3 - K & 2A^- & \frac{2(1+A_3)B^-}{1+B_3} \\ 2A^+ & -(2A_3+K) & \frac{2A^+B^-}{1+B_3} \\ \frac{2(1+A_3)B^+}{1+B_3} & \frac{2A^-B^+}{1+B_3} & K-2 \end{pmatrix}, \quad (11)$$

with $K = (1 + A_3)(1 - B_3)(1 + B_3)^{-1}$. The compatibility condition of this system gives the coupled spin equation hierarchy. As the particular example, let us consider the case when $N = 2$. Then the set of equations (9)-(10) takes the form

$$
\begin{aligned}
Y_x &= -i\lambda P Y, && (12)\\
Y_t &= \left(\lambda^2 V_2 + \lambda V_1\right) Y, && (13)
\end{aligned}
$$

where

$$
V_2 = -2iP, \quad V_1 = PP_x. \tag{14}
$$

The compatibility condition of the equations (12)-(13) gives the coupled spin equation (5)-(6).

LR type - II

The second type of LR for the coupled spin equation hierarchy can be written in the following form

$$Y_x = -i\lambda QY, \tag{15}$$

$$Y_t = \sum_{j=1}^{N} \lambda^j W_j Y. \tag{16}$$

Here

$$Q = Q_1 + Q_2, \tag{17}$$

where

$$Q_1 = \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ -A_1 & 0 & A_3 & -A_2 \\ -A_2 & -A_3 & 0 & A_1 \\ -A_3 & A_2 & -A_1 & 0 \end{pmatrix}, \qquad (18)$$

$$Q_2 = \begin{pmatrix} 0 & B_1 & B_2 & -B_3 \\ -B_1 & 0 & B_3 & B_2 \\ -B_2 & -B_3 & 0 & -B_1 \\ B_3 & -B_2 & B_1 & 0 \end{pmatrix}. \qquad (19)$$

From the compatibility condition of the set of equations (15)-(16) $Y_{xt} = Y_{tx}$ we obtain the coupled spin equation hierarchy.

Relation between solutions of the coupled spin equation and the Manakov system

Let A_j and B_j be the solution of the coupled spin equation (5)-(6). Then the solution of the Manakov system (1)-(2) is given by

$$q_1 = \frac{Re^{2i\nu}}{W(1+A_3)}, \tag{20}$$

$$q_2 = \frac{Ze^{2i\nu}}{W(1+A_3)}. \tag{21}$$

Here

$$W = 2 + \frac{(1+A_3)(1-B_3)}{1+B_3} = 2 + K,$$

$$R = WA^-_x - MA^-,$$

$$Z = W[(1+A_3)(1+B_3)^{-1}B^-]_x - M[(1+A_3)(1+B_3)^{-1}B^-].$$

$$M = A_{3x} + \frac{A^+A^-_x}{1+A_3} + \frac{A_{3x}(1-B_3)}{1+B_3} + \frac{(1+A_3)B^+B^-_x}{(1+B_3)^2} -$$

$$-\frac{(1+A_3)(1-B_3)B_{3x}}{(1+B_3)^2},$$

$$\nu = \partial_x^{-1}\left[\frac{A_1A_{2x} - A_{1x}A_2}{(1+A_3)W} - \frac{(1+A_3)(B_{1x}B_2 - B_1B_{2x})}{(1+B_3)^2 W}\right].$$

Gauge equivalence between the Γ-spin system and the Manakov system

In this section, we want to present the gauge equivalent counterpart of the Manakov system when $n = 3$. The Lax representation of the Manakov system (1)-(2) has the form

$$\Phi_x = U\Phi, \tag{22}$$

$$\Phi_t = V\Phi. \tag{23}$$

Here

$$U = -i\lambda\Sigma + U_0, \quad V = -2i\lambda^2\Sigma + 2\lambda U_0 + V_0 \tag{24}$$

with

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; U_0 = \begin{pmatrix} 0 & q_1 & q_2 \\ -\bar{q}_1 & 0 & 0 \\ -\bar{q}_2 & 0 & 0 \end{pmatrix}, \tag{25}$$

$$V_0 = i \begin{pmatrix} |q_1|^2 + |q_2|^2 & q_{1x} & q_{2x} \\ \bar{q}_{1x} & -|q_1|^2 & -\bar{q}_1 q_2 \\ \bar{q}_{2x} & -\bar{q}_2 q_1 & -|q_2|^2 \end{pmatrix}. \tag{26}$$

Let us now consider the gauge transformation

$$\Psi = g^{-1}\Phi, \quad g = \Phi_{\lambda=0}. \tag{27}$$

Then Ψ obeys the equations

$$\Psi_x = U'\Psi, \tag{28}$$

$$\Psi_t = V'\Psi, \tag{29}$$

where

$$U' = -i\lambda\Gamma, \quad V' = -2i\lambda^2\Gamma + \frac{1}{2}\lambda[\Gamma, \Gamma_x]. \tag{30}$$

Here

$$\Gamma = g^{-1}\Sigma g, \quad \Gamma^2 = I \tag{31}$$

and

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{pmatrix} \in su(3). \tag{32}$$

Elements of the Γ matrix satisfy some restrictions

$$\Gamma_{33} = -(1 + \Gamma_{11} + \Gamma_{22}), \quad \Gamma_{ij} = \bar{\Gamma}_{ji}, \tag{33}$$

and

$$\Gamma_{ik}\Gamma_{kj} + \Gamma_{i(k+1)}\Gamma_{(k+1)i} + \Gamma_{i(k+2)}\Gamma_{(k+2)i} = 0, (i \neq k \neq j), \tag{34}$$
$$\Gamma_{ik}\Gamma_{ki} + \Gamma_{i(k+1)}\Gamma_{(k+1)i} + \Gamma_{i(k+2)}\Gamma_{(k+2)i} = 1. \tag{35}$$

The compatibility condition of the equations (28)-(29) gives

$$i\Gamma_t + \frac{1}{2}[\Gamma, \Gamma_{xx}] = 0. \tag{36}$$

Relation between solutions of the coupled spin equation and the Γ-spin system

In the previous sections we have shown that to the one and same set of equations - the Manakov system (1)-(2), correspond two spin systems: the coupled spin equation (5)-(6) and the Γ-spin system (36). It tells us that between these two spin systems there must be some exact relation/correspondence. In other words, the 2-layer spin equation (5)-(6) and the Γ-spin system (36) are equivalent to each other by some exact transformations. Below we will present these transformations.

Direct transformation

According to the transformation, in terms of the spin vectors **A** and **B**, the elements of the Γ-spin system are expressed as

$$\Gamma = \frac{1}{2+K}\begin{pmatrix} 2A_3 - K & 2A^- & \frac{2(1+A_3)B^-}{1+B_3} \\ 2A^+ & -(2A_3+K) & \frac{2A^+B^-}{1+B_3} \\ \frac{2(1+A_3)B^+}{1+B_3} & \frac{2A^-B^+}{1+B_3} & K-2 \end{pmatrix}, \qquad (37)$$

where

$$K = \frac{(1+A_3)(1-B_3)}{1+B_3}. \qquad (38)$$

This is the direct transformation. This transformation allows us to find solutions of the Γ-spin system (36) if we know the solutions of the coupled spin equation (5)-(6).

Inverse transformation

According to the inverse transformation, solutions of the coupled spin equation can be expressed by the components of the Γ-spin system as

$$A = \frac{1}{1-\Gamma_{33}}\begin{pmatrix} \Gamma_{11}-\Gamma_{22} & 2\Gamma_{12} \\ 2\Gamma_{21} & \Gamma_{22}-\Gamma_{11} \end{pmatrix}, \tag{39}$$

$$B = \frac{1}{1-\Gamma_{22}}\begin{pmatrix} \Gamma_{11}-\Gamma_{33} & 2\Gamma_{13} \\ 2\Gamma_{31} & \Gamma_{33}-\Gamma_{11} \end{pmatrix}. \tag{40}$$

The transformations (39)-(40) is called the inverse transformation. Using the inverse transformation, we can find solutions of the coupled spin equation (5)-(6), if we know the solutions of the Γ-spin system (36).

Darboux transformation and exact solutions of the Γ-spin system

In this section, we construct the DT for the equation (36). To do this, let us consider the following transformation of solutions of the equations (28)-(29)

$$\Phi' = L\Psi, \tag{41}$$

where

$$L = \lambda N - I. \tag{42}$$

We require that Φ' satisfies the same Lax representation as (28)-(29) so that

$$\begin{aligned} \Phi'_x &= U'\Phi', & (43) \\ \Phi'_t &= V'\Phi', & (44) \end{aligned}$$

where $U' - V'$ depend on Γ' as $U - V$ on Γ.

The matrix L obeys the following equations

$$L_x + LU = U'L, \tag{45}$$

$$L_t + LV = V'L. \tag{46}$$

These equations yield the following equations for N

$$N_x = i\Gamma^{'} - i\Gamma, \tag{47}$$

$$N_t = -\Gamma'\Gamma'_x + \Gamma\Gamma_x \tag{48}$$

and

$$\Gamma^{'} = N\Gamma N^{-1}. \tag{49}$$

Also we have the following useful second form of the DT for $\Gamma^{'}$:

$$\Gamma^{'} = \Gamma - iN_x. \tag{50}$$

Let us consider the following set of equations

$$H_x = -i\Gamma H\Lambda, \tag{51}$$

$$H_t = -2i\Gamma H\Lambda^2 + \Gamma\Gamma_x H\Lambda, \tag{52}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{53}$$

$det\ H \neq 0$ and λ_k are complex constants. We now assume that the matrix N can be written as:

$$N = H\Lambda^{-1}H^{-1} = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}. \tag{54}$$

The inverse matrix we write as

$$N^{-1} = H\Lambda H^{-1} = \frac{1}{n}\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \tag{55}$$

or

$$N^{-1} = \frac{1}{n}\begin{pmatrix} n_{22}n_{33} - n_{23}n_{32} & -(n_{12}n_{33} - n_{13}n_{32}) & n_{12}n_{23} - n_{13}n_{22} \\ -(n_{21}n_{33} - n_{23}n_{31}) & n_{11}n_{33} - n_{31}n_{13} & -(n_{11}n_{23} - n_{21}n_{13}) \\ n_{32}n_{21} - n_{31}n_{22} & -(n_{11}n_{32} - n_{31}n_{12}) & n_{11}n_{22} - n_{21}n_{12} \end{pmatrix}$$

where $n = detN$ and has the form

$$n = n_{11}n_{22}n_{33} + n_{12}n_{23}n_{31} + n_{13}n_{32}n_{21} -$$

$$-n_{31}n_{22}n_{13} - n_{12}n_{21}n_{33} - n_{11}n_{23}n_{32}.$$

From these equations follow that N obeys the equations

$$N_x = iN\Gamma N^{-1} - i\Gamma, \tag{57}$$
$$N_t = \Gamma\Gamma_x - N\Gamma\Gamma_x N^{-1}, \tag{58}$$

which are equivalent to Eqs.(57)-(58) as we expected. The Γ and matrix solutions of the system (28)-(29) obey the condition

$$\Phi^{\dagger} = \Phi^{-1}, \quad \Gamma^{\dagger} = \Gamma, \tag{59}$$

which follow from the equations

$$\Phi^{\dagger}_x = i\lambda\Phi^{\dagger}\Gamma^{\dagger}, \quad (\Phi^{-1})_x = i\lambda\Phi^{-1}\Gamma^{-1}. \tag{60}$$

Here † denote an Hermitian conjugate. After some calculations we came to the formulas

$$H = \begin{pmatrix} \psi_1(\lambda_1; t, x, y) & \bar{\psi}_2(\lambda_1; t, x, y) & \bar{\psi}_3(\lambda_1; t, x, y) \\ \psi_2(\lambda_1; t, x, y) & -\bar{\psi}_1(\lambda_1; t, x, y) & 0 \\ \psi_3(\lambda_1; t, x, y) & 0 & -\bar{\psi}_1(\lambda_1; t, x, y) \end{pmatrix}, \quad (61)$$

where $\lambda_2 = \lambda_3 = \bar{\lambda}_1$,

$$H^{-1} = \frac{1}{\square\bar{\psi}_1} \begin{pmatrix} \bar{\psi}_1^2 & \bar{\psi}_1\bar{\psi}_2 & \bar{\psi}_1\bar{\psi}_3 \\ \bar{\psi}_1\psi_2 & -(|\bar{\psi}_1|^2 + |\psi_2|^2) & \psi_2\bar{\psi}_3 \\ \bar{\psi}_1\psi_3 & \bar{\psi}_2\psi_3 & -(|\bar{\psi}_1|^2 + |\psi_2|^2 \end{pmatrix}, \quad (62)$$

where

$$\square = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2. \quad (63)$$

So finally for the matrix N we get the following expression

$$N = \frac{1}{\Box}\begin{pmatrix} n_{11}\Box & n_{12}\Box & n_{13}\Box \\ \bar{\psi}_1\psi_2\epsilon_{12} & \frac{|\psi_2|^2}{\lambda_1} + \frac{|\bar{\psi}_1|^2+|\psi_3|^2}{\lambda_2} & \psi_2\bar{\psi}_3\epsilon_{12} \\ \bar{\psi}_1\psi_3\epsilon_{13} & \bar{\psi}_2\psi_3\epsilon_{13} & \frac{|\psi_3|^2}{\lambda_1} + \frac{|\psi_1|^2+|\psi_2|^2}{\lambda_3} \end{pmatrix}, \qquad (64)$$

where $\epsilon_{ij} = \lambda_i^{-1} - \lambda_j^{-1}$,

$$n_{11}\Box = \frac{|\psi_1|^2}{\lambda_1} + \frac{|\psi_2|^2}{\lambda_2} + \frac{|\psi_3|^2}{\lambda_3}, \qquad (65)$$

$$n_{12}\Box = \frac{\psi_1\bar{\psi}_2}{\lambda_1} - \frac{\psi_1\bar{\psi}_2}{\lambda_2} + \frac{\psi_2|\psi_3|^2}{\bar{\psi}_1}(\lambda_3^{-1} - \lambda_2^{-1}), \qquad (66)$$

$$n_{13}\Box = \frac{\psi_1\bar{\psi}_3}{\lambda_1} - \frac{\psi_1\bar{\psi}_3}{\lambda_3} + \frac{|\psi_2|^2\bar{\psi}_3}{\bar{\psi}_1}(\lambda_2^{-1} - \lambda_3^{-1}). \qquad (67)$$

Hence we can write the DT in terms of the eigenfunctions of the Lax representations (28)-(29) as

$$\Gamma^{[1]} =$$

$$\frac{1}{n}\begin{pmatrix} n_{11}m_{11} - n_{12}m_{21} - n_{13}m_{31} & n_{11}m_{12} - n_{12}m_{22} - n_{13}m_{32} & n_{11}m_{13} - n_{12}m \\ n_{21}m_{11} - n_{22}m_{21} - n_{23}m_{31} & n_{21}m_{12} - n_{22}m_{22} - n_{23}m_{32} & n_{21}m_{13} - n_{22}m \\ n_{31}m_{11} - n_{32}m_{21} - n_{33}m_{31} & n_{31}m_{12} - n_{32}m_{22} - n_{33}m_{32} & n_{31}m_{13} - n_{32}m \end{pmatrix}$$

To construct the 1-soliton solution of the Γ-spin system (36), now we consider a seed solution

$$\Gamma^{[0]} = \Sigma. \tag{69}$$

In our case the eigenfunctions are given by

$$\psi_1^{[0]} = e^{-\theta+i\delta_1}, \quad \psi_2^{[0]} = e^{\theta+i\delta_2}, \quad \psi_3^{[0]} = e^{\theta+i\delta_3}, \tag{70}$$

where δ_i are complex constants and

$$\theta = \theta_1 + i\theta_2 = -i\lambda_1 x - 2i\lambda_1^2 t. \tag{71}$$

Then we get

$$\Gamma^{[1]} = \begin{pmatrix} \Gamma_{11}^{[1]} & \Gamma_{12}^{[1]} & \Gamma_{13}^{[1]} \\ \Gamma_{21}^{[1]} & \Gamma_{22}^{[1]} & \Gamma_{23}^{[1]} \\ \Gamma_{31}^{[1]} & \Gamma_{32}^{[1]} & \Gamma_{33}^{[1]} \end{pmatrix}, \tag{72}$$

where n_{ij} and m_{ij} are given by the equations (65)-(67).

The 1-soliton solutions of the coupled spin equation

Let us now we present the formulas for the 1-soliton solution of the coupled spin equation (5)-(6). Its seed solution we write as

$$A^{[0]} = \sigma_3, \quad B^{[0]} = \sigma_3. \tag{73}$$

To find the 1-soliton solution of the coupled spin equation here we use the inverse M-transformation . The inverse transformation allows us to find solutions of the coupled spin equation (5)-(6), if we know the solutions of the Γ-spin system (36). So the 1-soliton solution of the coupled spin equation has the form

$$
A^{[1]} = \frac{1}{1-\Gamma_{33}^{[1]}} \begin{pmatrix} \Gamma_{11}^{[1]} - \Gamma_{22}^{[1]} & 2\Gamma_{12}^{[1]} \\ 2\Gamma_{21}^{[1]} & \Gamma_{22}^{[1]} - \Gamma_{11}^{[1]} \end{pmatrix}, \tag{74}
$$

$$
B^{[1]} = \frac{1}{1-\Gamma_{22}^{[1]}} \begin{pmatrix} \Gamma_{11}^{[1]} - \Gamma_{33}^{[1]} & 2\Gamma_{13}^{[1]} \\ 2\Gamma_{31}^{[1]} & \Gamma_{33}^{[1]} - \Gamma_{11}^{[1]} \end{pmatrix}, \tag{75}
$$

where $\Gamma_{ij}^{[1]}$ are given by the formulas (72) and (68).

Conclusion

In this paper we have presented the DT for the Γ-spin system which is the integrable $su(3)$-valued spin system. In particular, we have given the explicit formula for its 1-soliton solution. Then we have shown how construct soliton solutions of the coupled spin equation for the two coupled $su(2)$ - valued spin systems. For this purpose we have used the DT formulas of the Γ-spin system. Also the Lax representation of the coupled spin equation is presented. Using this Lax representation, the gauge equivalence between the coupled spin equation and the Manakov system is established. The results obtained in this paper will be useful in the study of nonlinear dynamics of multilayer magnetic systems. Also they will be useful in differential geometry of curves and surfaces to find the integrable deformations of interacting curves and surfaces.

References

[1] N. A. Kostov, R. Dandoloff, V. S. Gerdjikov, G. G. Grahovski. *The Manakov system as two moving interacting curves*, In the Proceedings of the International Workshop "Complex structures and vector fields", August 21–26, 2006, Sofia, Bulgaria. Eds.: K. Sekigawa, S. Dimiev. World Scientific (2007).
[2] C. Senthilkumar, M. Lakshmanan, B. Grammaticos, A. Ramani, Phys. Lett. A **356** 339-345 (2006).
[3] Myrzakulov R., Rahimov F.K., Serikbaev N.S. *On the geometry of stationary Heisenberg ferromagnets*. In: "Non-linear waves: Classical and Quantum Aspects", Kluwer Academic Publishers, Dordrecht, Netherlands, P. 543-549 (2004).
[4] Z.S. Yersultanova, M. Zhassybayeva, K. Yesmakhanova, G. Nugmanova, R. Myrzakulov. International Journal of Geometric Methods in Modern Physics, **13**, N1, 1550134 (2016).
[5] Chen Hai, Zhou Zi-Xiang. *Darboux Transformation with a Double Spectral Parameter for the Myrzakulov-I Equation*. Chin. Phys. Lett., **31**, N12, 120504 (2014).

Thank you for attention!