Deformations Without Bending: Explicit Examples

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Geometry, Integrability and Quantization
June 2-7, 2018

Outline

Deformations Without Bending: Explicit Examples

Flugge (1960), Novozhilov (1962)

Shells are walls (in the widest sense of the word)

Diversity of shells: wall of a tank, metal hull of airplane, rubber hull of a balloon, soap bubble, surface of a liquid

The thickness of a shell is very small compared to other dimensions.

Geometrically the shell is described by the shape of its middle surface and its thickness. Mechanically the shell is described by the field of stresses and stress resultants (forces and moments).

Shells are capable of transmitting loads from one part to another part of the shell.

The consequent deformations are described by strains and displacements.

loads $\longrightarrow$ stress resultants $\longrightarrow$ strains $\longrightarrow$ displacements

equilibrium conditions (1st arrow)

Hooke's elastic law (2nd)

geometrical connections (3rd)

Shells of revolution: tanks, pressure vessels, domes

Shell element is cut out by two meridians and two parallels.

three equations of equilibrium

three unknown stress resultants N_θ, N_φ, $N_{\theta\varphi}$

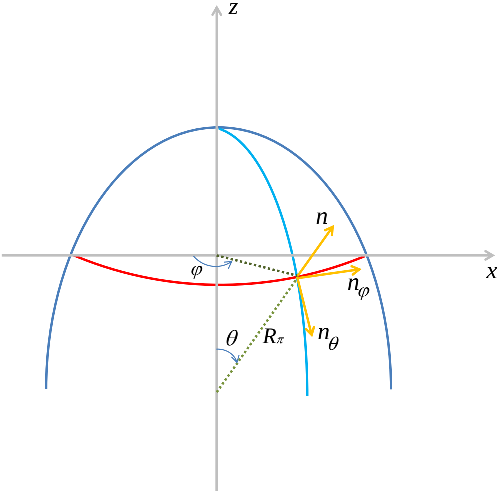
bending and twisting moments are neglected

meridional force N_θ

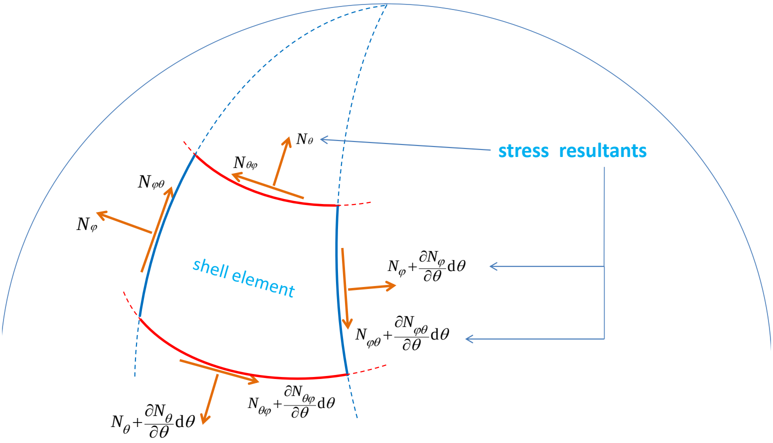
hoop force N_φ

shearing force $N_{\theta\varphi}$

Middle Surface of a Shell of Revolution



Shell Element is cut out by two meridians and two parallels



Non-Bending Condition
(Gurevich and Kalinin, 1981)

free of bending deformations = normals do not turn

shell under constant pressure
perpendicular to the middle surface

$$\left(3 - \frac{R_\pi}{R_\mu}\right)\frac{\mathrm{d}R_\pi}{\mathrm{d}\theta} - R_\pi \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_\pi}{R_\mu}\right) = 0$$

meridional R_μ and parallel (hoop) R_π principal radii
angle θ between the normal and the axes of revolution

Non-Bending Condition

$$\left(3 - \frac{R_\pi}{R_\mu}\right)\frac{\mathrm{d}R_\pi}{\mathrm{d}\theta} - R_\pi \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R_\pi}{R_\mu}\right) = 0$$

right circular cylinder: $R_\mu = \infty,\ R_\pi = \text{const}$

sphere: $R_\mu = R_\pi = \text{const}$

linear Weingarten surface $LW(2)$: $R_\mu = \frac{1}{3}R_\pi$

Non-Bending Condition
in terms of principal curvatures k_μ, k_π

$$k_\mu = 2a\, k_\pi^2 + 3k_\pi, \quad a = \text{const}$$

right circular cylinder: $a = -\frac{3}{2k_\pi} = \text{const}$

sphere: $a = -\frac{1}{k_\pi} = \text{const}$

linear Weingarten surface $LW(2)$: $a = 0,\ k_\mu = 3k_\pi$

Three Non-Bending Surfaces

(1) right circular cylinder

(2) sphere

(3) $LW(2)$-balloon

What is a $LW(2)$-balloon?

$LW(n)$-Surfaces of Revolution
(Pulov, Hadzhilazova and Mladenov, 2018)

Surfaces of revolution whose principal curvatures obey a linear relation

$$k_\mu = (n+1)k_\pi, \qquad n = 0,\, 1,\, 2, \ldots$$

are referred to as $LW(n)$-surfaces.

$LW(n)$-Surfaces of Revolution

$LW(0)$ Sphere ($k_\mu = k_\pi$)

$LW(1)$ Mylar Balloon ($k_\mu = 2k_\pi$)

$LW(2)$-Balloon ($k_\mu = 3k_\pi$)

Variational Characterization of $LW(n)$
(Mladenov and Oprea, 2007)

Find a profile curve $z = f(x)$ of a surface of revolution

by extremizing the functional $\quad J_n(f) = \int\limits_0^r x^n f(x)\,\mathrm{d}x, \quad n = 0, 1, \dots$

subject to a fixed profile arclength $\quad \int\limits_0^r \sqrt{1 + f'(x)^2}\,\mathrm{d}u = \mathrm{const} > 0$

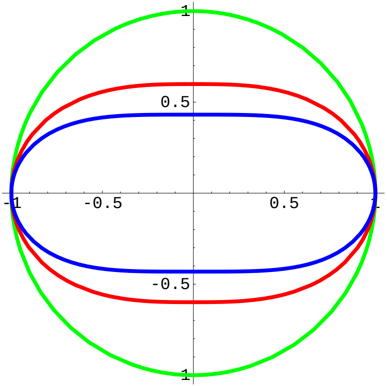
$LW(n)$-Surfaces of Revolution

$LW(0)$ Sphere – maximum area $J_0(f)$ of the meridional section

$LW(1)$ Mylar Balloon – maximum volume $J_1(f)$

$LW(2)$-Balloon $(k_\mu = 3k_\pi)$ – extremal value of $J_2(f)$

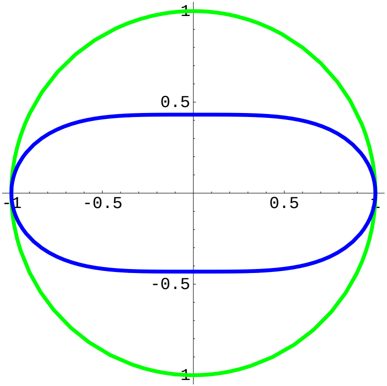
Sphere, Mylar Balloon, $LW(2)$-Balloon



$LW(n)$-Balloons (non-bending)

$k_\mu = (n+1)k_\pi, \; n = 0$, 1 and 2

Sphere $\qquad$ $LW(2)$-Balloon



What shape do the rest of the non-bending balloons have?
(for arbitrary $a \in \mathbb{R}$)

$$k_\pi = \frac{f'}{x(1+f'^2)^{1/2}}, \qquad k_\mu = \frac{\mathrm{d}(x\, k_\pi)}{\mathrm{d}x}$$

$$k_\mu = 2a\, k_\pi^2 + 3k_\pi\,, \qquad a = \text{const}$$

three unknown functions $f(x)$, $k_\mu(x)$, $k_\pi(x)$
$z = f(x)$ – the profile curve of the shell

Supporting parallel

The supporting parallel lies on the ground (in the XOY plane) and bears the load of the shell.

r – radius of the supporting parallel

r_μ, r_π – meridional and parallel principal radii of the supporting parallel

boundary conditions – $k_\pi(r) = \frac{1}{r_\pi}, \quad k_\mu(r) = \frac{1}{r_\mu}, \quad f(r) = 0$

$\mathring{\theta}$ – angle at which the shell meets the ground: $r = r_\pi \sin \mathring{\theta}$

The Principal Curvatures

$$k_\mu(x) = \frac{\left[3r_\pi^2(a+r_\pi) - ax^2\mathrm{cosec}^2\mathring{\theta}\right] x^2\mathrm{cosec}^2\mathring{\theta}}{\left[r_\pi^2(a+r_\pi) - ax^2\mathrm{cosec}^2\mathring{\theta}\right]^2}$$

$$k_\pi(x) = \frac{x^2\mathrm{cosec}^2\mathring{\theta}}{r_\pi^2(a+r_\pi) - ax^2\mathrm{cosec}^2\mathring{\theta}}$$

$\mathring{\theta}$ – angle at which the shell meets the ground

$a = \frac{r_\pi^2 - 3r_\pi r_\mu}{2r_\mu}$, (r_μ, r_π) – two free parameters

Upper Right Branch
(two parametrical family)

$$f(x) = \int_x^r \frac{\tau^3 \mathrm{d}\tau}{\sqrt{(ar^2 + r^2 r_\pi - a\tau^2)^2 - \tau^6}}, \qquad 0 \le x \le r$$

$$a = \frac{r_\pi^2 - 3r_\pi r_\mu}{2r_\mu}, \qquad (r_\mu, r_\pi) \text{ -- two free parameters}$$

r – radius of the supporting parallel

Upper Right Branch
(new integration variable)

$$f(x) = \nu r \int_{(x/r)^2}^{1} \frac{t\mathrm{d}t}{\sqrt{c^2(1-\nu+(3\nu-1)t)^2 - 4\nu^2 t^3}}, \qquad 0 \le x \le r$$

$$\nu = \frac{r_\mu}{r_\pi}, \quad c = \frac{r_\pi}{r} = \operatorname{cosec} \mathring{\theta}, \quad (r_\mu, r_\pi) - \text{two free parameters}$$

two parametrical family of shells of revolution
that deform without bending under uniform pressure

Upper Right Branch

$$\mathring{\theta} = \frac{\pi}{2}, \quad r_\pi = r, \quad \nu = \frac{r_\mu}{r}$$

$$f(x) = \nu r \int_{(x/r)^2}^{1} \frac{t\,\mathrm{d}t}{\sqrt{(1-t)((1-\nu)^2 - (1-\nu)(1-5\nu)t + 4\nu^2 t^2)}}, \qquad 0 \le x \le r$$

$\mathring{\theta}$ – angle at which the shell meets the ground

r – radius of the supporting parallel

ν – free parameter

Upper Half of the Shell

$\mathring{\theta} = \frac{\pi}{2}, \quad 0 \leq \nu \leq \frac{1}{9}$

$$x(u,v) = r \sin u \cos v, \quad y(u,v) = r \sin u \sin v$$

$$z(u) = r\left(\frac{1}{\lambda} F(\varphi(u),k)) + \lambda E(\varphi(u),k)) - \lambda \tan\varphi(u)\sqrt{1 - k^2 \sin^2\varphi(u)}\right)$$

where

$$\varphi(u) = \arcsin\left(\frac{2\sqrt{2}\nu\cos u}{\sqrt{(5\nu+\sigma-1)(\nu-1) - 8\nu^2\sin^2 u}}\right), \quad k = \frac{\sqrt{\sigma(1-\nu)}}{2\lambda\nu}$$

$$\lambda = \frac{1}{2\nu}\sqrt{\frac{(1-\nu)\sigma - 3\nu^2 - 6\nu + 1}{2}}, \quad \sigma = \sqrt{1 - 10\nu + 9\nu^2}$$

$$u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad v \in [0, 2\pi]$$

$F(\varphi,k)$, $E(\varphi,k)$ – incomplete elliptic integrals

Upper Half of the Shell

$$\mathring{\theta} = \tfrac{\pi}{2}, \ \ \tfrac{1}{9} \leq \nu \leq 1$$

$$x(u,v) = r \sin u \, \cos v, \quad y(u,v) = r \sin u \, \sin v$$

$$z(u) = \frac{r}{\sqrt[4]{\nu}} \left(\frac{\sqrt{\nu}-1}{2} F(\varphi(u),k)) + E(\varphi(u),k)) - \frac{\sin\varphi(u)\sqrt{1-k^2\sin^2\varphi(u)}}{1+\cos\varphi(u)} \right)$$

where

$$\varphi(u) = \arccos\left(\frac{1-\sqrt{\nu}\cos^2 u}{1+\sqrt{\nu}\cos^2 u}\right), \quad k = \frac{\sqrt{8\nu^{3/2}+3\nu^2+6\nu-1}}{4\nu^{3/4}}$$

$$u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad v \in [0, 2\pi]$$

$F(\varphi,k)$, $E(\varphi,k)$ – incomplete elliptic integrals

Right Hand Side of the Profile

$\mathring{\theta} = \frac{\pi}{2}, \; 0 \leq \nu \leq 1$

$$x(u) = \sqrt{\lambda\,\wp(u;\, g_2, g_3) + \sigma}, \qquad z(u) = \frac{\lambda}{2}[\sigma\, u - \lambda\,\zeta(u;\, g_2, g_3)]$$

where

$$g_2 = \frac{r^4(1-3\nu)(1-9\nu+3\nu^2-3\nu^3)}{24\sqrt[3]{2}\,\nu^4}$$

$$g_3 = \frac{r^6(3\nu^2+6\nu-1)(1-12\nu+30\nu^2-36\nu^3+9\nu^4)}{864\,\nu^6}$$

$$\lambda = -\sqrt[3]{4}, \quad \sigma = \frac{r^2(1-3\nu)^2}{12\,\nu^2}, \quad u \in [0, \pi],$$

$\wp(u;\, g_2, g_3)$, $\zeta(u;\, g_2, g_3)$ – Weierstrass Functions

First Fundamental Form

$0 \leq \nu \leq 1$

$$E = \frac{r^2(1+\nu+(1-3\nu)\cos 2u)^2}{2(1+2\nu+(1-6\nu+\nu^2)\cos 2u+\nu^2\cos 4u)}$$

$$F = 0$$

$$G = r^2 \sin^2 u \qquad u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Second Fundamental Form

$0 \leq \nu \leq 1$

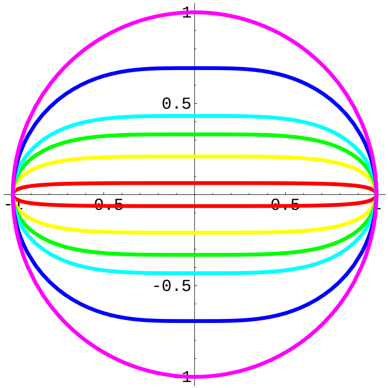
$$L = \frac{2r\nu(5 - 3\nu - (3\nu - 1)\cos 2u)\sin^2 u}{1 + 2\nu + (1 - 6\nu + \nu^2)\cos 2u + \nu^2 \cos 4u}$$

$$M = 0$$

$$N = \frac{4r\nu \sin^4 u}{1 + \nu - (3\nu - 1)\cos 2u} \qquad u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

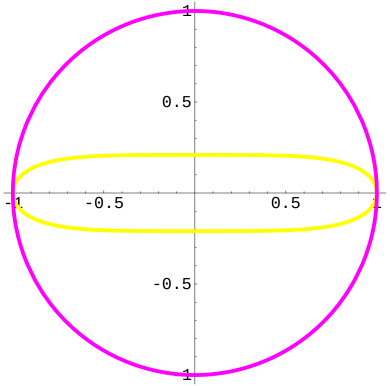
Profiles of the Middle Surfaces of Non-Bending Shells of Revolution Under Uniform Pressure

$r_\mu = 1, \quad 2/3, \quad 1/3, \quad 2/9, \quad 1/9, \quad 1/50$



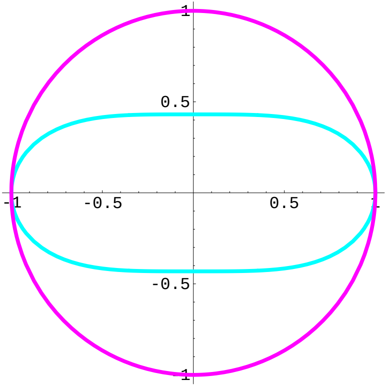
Profiles of the Middle Surfaces of Non-Bending Shells of Revolution Under Uniform Pressure

$r_\mu = \underline{\underline{\mathbf{1}}},\quad 2/3,\quad 1/3,\quad 2/9,\quad \underline{\underline{1/9}},\quad 1/50$



Profiles of the Middle Surfaces of Non-Bending Shells of Revolution Under Uniform Pressure

$r_\mu = \underline{\underline{1}},\quad 2/3,\quad \underline{\underline{1/3}},\quad 2/9,\quad 1/9,\quad 1/50$



Surface Areas of Non-Bending Shells

$$
\begin{aligned}
A(S) &= 2\int_0^{\pi/2}\int_0^{2\pi}\sqrt{EG-F^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= 2r^2\int_0^{\pi/2}\int_0^{2\pi}\frac{(1+\nu+(1-3\nu)\cos 2u)\sin u}{\sqrt{2(1+2\nu+(1-6\nu+\nu^2)\cos 2u+\nu^2\cos 4u)}}\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

r – radius of the supporting parallel

$\nu = r_\mu/r$ – free parameter

Surface Areas of Non-Bending Shells (new integration variable)

$$A(S) = \frac{r^2}{2\sqrt{2}\,\nu} \int_{-1}^{1} \int_{0}^{2\pi} \frac{1+\nu+(1-3\nu)t}{\sqrt{(t+1)(t-b)(t-c)}}\,\mathrm{d}t\,\mathrm{d}v$$

$$b = \frac{-1+6\nu-\nu^2+\sqrt{(\nu-1)^3\,(9\nu-1)}}{4\nu^2} \qquad c = \frac{-1+6\nu-\nu^2-\sqrt{(\nu-1)^3\,(9\nu-1)}}{4\nu^2}$$

$A(S)$ – elliptic integral

r – radius of the supporting parallel

$\nu = \frac{r_\mu}{r}$ – free parameter

Bolshoy Ice Dome, Sochi, Russia, 2012

Looks like LW(2)-balloon, isn't it?

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