On Reductions of a Matrix Generalized Heisenberg Ferromagnet Equation

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Introduction

- Classical $1+1$-dimensional Heisenberg ferromagnet equation (HF)

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \qquad \mathbf{S}^2 = 1.$$

  $\mathbf{S} = (S_1, S_2, S_3)$ is the spin vector of a one-dimensional ferromagnet.
- HF has a zero curvature representation $[L(\lambda), A(\lambda)] = 0$ with (Lax) operators $L(\lambda)$ and $A(\lambda)$ of the form:

$$\begin{aligned} L(\lambda) &= \mathrm{i}\partial_x - \lambda S, \qquad \lambda \in \mathbb{C}, \\ A(\lambda) &= \mathrm{i}\partial_t + \frac{\mathrm{i}\lambda}{2}[S, S_x] + 2\lambda^2 S \end{aligned}$$

  where $\mathrm{i} = \sqrt{-1}$ and

$$S = \begin{pmatrix} S_3 & S_1 - \mathrm{i}S_2 \\ S_1 + \mathrm{i}S_2 & -S_3 \end{pmatrix}.$$

- Integrable generalizations of HF:
  - Two component system [Gerdjikov-Mikhailov-Valchev] and [Yanovski-Valchev]:

$$
\begin{aligned}
&\mathrm{i}u_{1,t} + u_{1,xx} + [(\epsilon u_1 u_{1,x}^* + u_2 u_{2,x}^*)u_1]_x = 0, \qquad \epsilon^2 = 1,\\
&\mathrm{i}u_{2,t} + u_{2,xx} + [(\epsilon u_1 u_{1,x}^* + u_2 u_{2,x}^*)u_2]_x = 0
\end{aligned}
$$

where u_1 and u_2 satisfy the constraint:

$$\epsilon|u_1|^2 + |u_2|^2 = 1.$$

  - Vector system [Golubchik-Sokolov]:

$$
\begin{aligned}
&\mathrm{i}\mathbf{u}_t + \left[\left(\mathbf{u}\mathbf{v}^T\right)_x \mathbf{u}\right]_x + 4\left(\mathbf{u}^T K \mathbf{v}\right)\mathbf{u} = 0,\\
&\mathrm{i}\mathbf{v}_t - \left[\left(\mathbf{v}\mathbf{u}^T\right)_x \mathbf{v}\right]_x - 4\left(\mathbf{u}^T K \mathbf{v}\right)\mathbf{v} = 0.
\end{aligned}
$$

K is a constant diagonal matrix and the vectors $\mathbf{u}$ and $\mathbf{v}$ fulfill:

$$\mathbf{u}^T\mathbf{v} = 1.$$

- **Purpose of the talk:** Discussion of a new integrable matrix generalizations of HF and its hierarchy (work in progress).
- Main object of study is the system:

$$\begin{aligned} \mathrm{i}\mathbf{u}_t + \left[(\mathbf{u}\mathbf{v}^T)_x\mathbf{u} - \mathbf{u}(\mathbf{v}^T\mathbf{u})_x\right]_x &= 0, \\ \mathrm{i}\mathbf{v}_t + \left[\mathbf{v}(\mathbf{u}^T\mathbf{v})_x - (\mathbf{v}\mathbf{u}^{\mathbf{T}})_x\mathbf{v}\right]_x &= 0 \end{aligned}$$

  for the $n \times m$ matrices $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$.
- Pseudo-Hermitian reduction: $\mathbf{v} \sim \mathbf{u}^*$.

Matrix Heisenberg Ferromagnet Equation

- Lax pair related to $\mathrm{SU}(m+n)/\mathrm{S}(\mathrm{U}(m)\times\mathrm{U}(n))$
  Consider the following $L-A$ pair:

$$\begin{aligned} L(\lambda) &:= \mathrm{i}\partial_x - \lambda S, \qquad \lambda\in\mathbb{C} \\ A(\lambda) &:= \mathrm{i}\partial_t + \lambda A_1 + \lambda^2 A_2 \end{aligned}$$

  where

$$\begin{aligned} S &:= \begin{pmatrix} 0 & \mathbf{u}^T \\ \mathbf{v} & 0 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & \mathbf{a}^T \\ \mathbf{b} & 0 \end{pmatrix}, \\ A_2 &:= \frac{2r}{m+n}\mathbb{1}_{m+n} - S^2, \qquad r\leq min(m,n). \end{aligned}$$

  Above, $\mathbf{u}(x,t)$, $\mathbf{v}(x,t)$, $\mathbf{a}(x,t)$ and $\mathbf{b}(x,t)$ are $n\times m$ matrices.

- Additional algebraic constraint
  We require that
$$S^3 = S.$$

- Spectral properties of S
  The above constraint means that S is diagonalizable with eigenvalues 0 (multiplicity $m + n - 2r$) and ± 1 (multiplicity r).

- Characteristic polynomial of ad_S (technical remark)
  As a result of the above constraints we have:
$$\mathrm{ad}_S^5 - 5\mathrm{ad}_S^3 + 4\mathrm{ad}_S = 0.$$
  This is why we can pick up
$$\mathrm{ad}_S^{-1} := \frac{1}{4}(5\mathrm{ad}_S - \mathrm{ad}_S^3)$$
  as an (right) inverse operator of ad_S.

- Additional algebraic constraints II
  Written in more detail, the constraint $S^3 = S$ reads:

$$\mathbf{u}^T \mathbf{v} \mathbf{u}^T = \mathbf{u}^T, \qquad \mathbf{v} \mathbf{u}^T \mathbf{v} = \mathbf{v}.$$

Remark

The above relations mean that

$$\left(\mathbf{u}^T \mathbf{v}\right)^2 = \mathbf{u}^T \mathbf{v}, \qquad \left(\mathbf{v} \mathbf{u}^T\right)^2 = \mathbf{v} \mathbf{u}^T.$$

The two projectors have the same rank $r \leq min(m, n)$.

- Special cases:
  - Assume that $m < n$. Then both constraints can be replaced with

$$\mathbf{u}^T\mathbf{v} = \mathbb{1}_m.$$

In particular, if $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ are n-vectors we have:

$$\mathbf{u}^T\mathbf{v} = 1.$$

  - When $m > n$ we can replace the constraints with

$$\mathbf{v}\mathbf{u}^T = \mathbb{1}_n.$$

  - For $m = n$ ($\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ are square matrices) either of the above special algebraic constraints lead to a trivial flow. In this case one needs the more general algebraic constraint

$$\mathbf{u}^T\mathbf{v}\mathbf{u}^T = \mathbf{u}^T, \qquad \mathbf{v}\mathbf{u}^T\mathbf{v} = \mathbf{v}.$$

- Pseudo-Hermitian reduction conditions
  The Lax pair above is subject to

$$HL(-\lambda)H = L(\lambda), \qquad HA(-\lambda)H = A(\lambda),$$

for $H = \mathrm{diag}\,(-\mathbb{1}_m, \mathbb{1}_n)$. If we impose an extra reduction

$$\mathcal{E}_{m+n}S^{\dagger}\mathcal{E}_{m+n} = S, \qquad \mathcal{E}_{m+n}A^{\dagger}_{1,2}\mathcal{E}_{m+n} = A_{1,2}$$

where

$$\begin{aligned} \mathcal{E}_{m+n} &= \mathrm{diag}\,(\mathcal{E}_m, \mathcal{E}_n), \qquad \mathcal{E}_m = \mathrm{diag}\,(\epsilon_1, \epsilon_2, \ldots, \epsilon_m), \\ \mathcal{E}_n &= \mathrm{diag}\,(\epsilon_{m+1}, \epsilon_{m+2}, \ldots, \epsilon_{m+n}), \quad \epsilon_j^2 = 1, \quad j = 1, \ldots, m+n \end{aligned}$$

then we immediately have

$$\mathbf{v} = \mathcal{E}_n\mathbf{u}^*\mathcal{E}_m, \qquad \mathbf{b} = \mathcal{E}_n\mathbf{a}^*\mathcal{E}_m.$$

That reduction condition is called pseudo-Hermitian.

- The zero curvature condition $[L(\lambda), A(\lambda)] = 0$ leads to the connections:

$$\begin{aligned} \mathbf{a} &= \mathrm{i}\left(\mathbf{u}(\mathbf{v}^T\mathbf{u})_x - (\mathbf{u}\mathbf{v}^T)_x\mathbf{u}\right), \\ \mathbf{b} &= \mathrm{i}\left((\mathbf{v}\mathbf{u}^T)_x\mathbf{v} - \mathbf{v}(\mathbf{u}^T\mathbf{v})_x\right) \end{aligned}$$

and the matrix system:

$$\begin{aligned} \mathrm{i}\mathbf{u}_t + \left[(\mathbf{u}\mathbf{v}^T)_x\mathbf{u} - \mathbf{u}(\mathbf{v}^T\mathbf{u})_x\right]_x &= 0, \\ \mathrm{i}\mathbf{v}_t + \left[\mathbf{v}(\mathbf{u}^T\mathbf{v})_x - (\mathbf{v}\mathbf{u}^{\mathbf{T}})_x\mathbf{v}\right]_x &= 0. \end{aligned}$$

- Examples

Example

Let us consider the case when $m = 1$ and $n \geq 2$, i.e. $\mathbf{u}$ is a n-component vector function.
Without any loss of generality we can set $\mathcal{E}_1 = 1$ and assume that at least one diagonal entry of $\mathcal{E}_n$ is 1. Then generalized HF acquires the following form:

$$\mathrm{i}\mathbf{u}_t + \mathbf{u}_{xx} + \left(\mathbf{u}\mathbf{u}_x^{\dagger}\mathcal{E}_n\mathbf{u}\right)_x = 0$$

where $\mathbf{u}$ must satisfy

$$\mathbf{u}^T\mathcal{E}_n\mathbf{u}^* = 1.$$

That relation represents geometrically a sphere embedded in $\mathbb{R}^{2n}$ provided $\mathcal{E}_n = \mathbb{1}_n$ and a hyperboloid in $\mathbb{R}^{2n}$ otherwise.

Integrable Hierarchy and Recursion Operators

- General flow Lax pair
  Let us consider the following L-A pair:

$$L(\lambda) := \mathrm{i}\partial_x - \lambda S,$$
$$A(\lambda) := \mathrm{i}\partial_t + \sum_{j=1}^{N} \lambda^j A_j, \qquad N \geq 2.$$

- Recurrence relations
  The condition $[L(\lambda), A(\lambda)] = 0$ gives rise to:

$$[S, A_N] = 0,$$
$$\dots$$
$$\mathrm{i}\partial_x A_k - [S, A_{k-1}] = 0, \qquad k = 2, \dots, N,$$
$$\dots$$
$$\partial_x A_1 + \partial_t S = 0.$$

- Splitting of the coefficients
  In order to resolve the recurrence relations we apply the following "adapted"splitting

$$A_j = A_j^{\mathrm{a}} + A_j^{\mathrm{d}}, \qquad j = 1, \ldots, N$$

of the coefficients of $A(\lambda)$, i.e. a splitting such that

$$[S, A_j^{\mathrm{d}}] = 0$$

It is easily seen that

$$A_N^{\mathrm{a}} = 0.$$

Taking into account the constraint $S^3 = S$ one picks up

$$A_N = \begin{cases} c_N S, & N \equiv 1 \pmod 2 \\ c_N S_1, & N \equiv 0 \pmod 2 \end{cases}, \qquad c_N \in \mathbb{R}$$

where

$$S_1 = S^2 - \frac{2r}{m+n} \mathbb{1}_{m+n}, \quad r \leq min(m, n).$$

- Resolving the recurrence relations through generating operators

$$A^{\mathrm{a}}_{j-1} = \begin{cases} \Lambda A^{\mathrm{a}}_{j} + \mathrm{i} c_j \mathrm{ad}\,_S^{-1} S_{1,x}, & j \equiv 0 \pmod 2 \\ \Lambda A^{\mathrm{a}}_{j} + \mathrm{i} c_j \mathrm{ad}\,_S^{-1} S_{x}, & j \equiv 1 \pmod 2 \end{cases}, \qquad c_j \in \mathbb{R}$$

where

$$\begin{aligned} \Lambda \quad := \quad & \mathrm{i}\mathrm{ad}\,_S^{-1} \left\{ \partial_x(.)^{\mathrm{a}} - \frac{S_x}{2r} \partial_x^{-1} \mathrm{tr}\, \left[S(\partial_x(.))^{\mathrm{d}} \right] \right. \\ & \left. - \frac{(m+n) S_{1,x}}{2r(m+n-2r)} \partial_x^{-1} \mathrm{tr}\, \left[S_1(\partial_x(.))^{\mathrm{d}} \right] \right\}. \end{aligned}$$

The symbol ∂_x^{-1} stand for the (formal) right inverse operators of ∂_x and

$$\mathrm{ad}\,_S^{-1} := \frac{1}{4}(5\mathrm{ad}\,_S - \mathrm{ad}\,_S^3).$$

The operator Λ^2 is called recursion (generating) operator.

- Description of the integrable hierarchy
  An arbitrary member of the integrable hierarchy can be written down as follows:

$$\mathrm{iad}\,_S^{-1} S_t + \sum_k c_{2k}\Lambda^{2k} S_1 + \sum_k c_{2k-1}\Lambda^{2k-1} S = 0.$$

  where we have extended the action of Λ on the S-commuting part by requiring

$$\Lambda S := \mathrm{iad}\,_S^{-1} S_x, \qquad \Lambda S_1 := \mathrm{iad}\,_S^{-1} S_{1,x}.$$

Recursion Operators

- Gürses-Karasu-Sokolov method
  An alternative approach to find recursion operators is Gürses-Karasu-Sokolov method. In order to see how it works, let us consider the (original) Lax representations:

$$\begin{aligned} \mathrm{i}L_\tau &= [L, \tilde{V}], \\ \mathrm{i}L_t &= [L, V] \end{aligned}$$

where

$$L(\lambda) = \mathrm{i}\partial_x - \lambda S, \qquad S = \begin{pmatrix} 0 & \mathbf{u}^T \\ \mathbf{v} & 0 \end{pmatrix}, \qquad \lambda \in \mathbb{C}$$

and

$$V(x,t,\lambda) = \sum_{k=1}^{N} \lambda^k A_k(x,t), \qquad \tilde{V}(x,t,\lambda) = \sum_{k=1}^{N+2} \lambda^k \tilde{A}_k(x,t)$$

are two adjacent flows with evolution parameters t and τ respectively.

- Interrelation between V and $\tilde{V}$
  Due to the condition

$$HV(-\lambda)H = V(\lambda), \qquad H\tilde{V}(-\lambda)H = \tilde{V}(\lambda), \qquad H = \mathrm{diag}\,(-\mathbb{1}_m, \mathbb{1}_n),$$

  the flows V and $\tilde{V}$ are interrelated in the following way:

$$\tilde{V}(x,t,\lambda) = \lambda^2 V(x,t,\lambda) + B(x,t,\lambda).$$

- Recurrence relations of Lax representation
  After substituting the above relations in the Lax representation, we obtain:

$$\mathrm{i}L_\tau = \mathrm{i}\lambda^2 L_t + [L, B].$$

  The remainder B is sought in the form (this implies $N = 2$):

$$B(x,t,\lambda) = \lambda^2 B_2(x,t) + \lambda B_1(x,t).$$

After substituting the explicit expression for B, we get the recurrence relations:

$$\begin{aligned} \mathrm{i}\partial_t S + [S, B_2] &= 0, \\ \mathrm{i}\partial_x B_2 - [S, B_1] &= 0, \\ \partial_\tau S + \partial_x B_1 &= 0 \end{aligned}$$

which are resolved to give

$$S_\tau = \mathrm{ad}_S \Lambda^2 \mathrm{ad}_S^{-1} S_t.$$

Since we may also define the recursion operator $\mathcal{R}$ as

$$S_\tau = \mathcal{R} S_t,$$

we immediately see that

$$\mathcal{R} = \mathrm{ad}_S \Lambda^2 \mathrm{ad}_S^{-1}.$$

Conclusion

- We have introduced a matrix system containing all the known models generalizing the classical HF. As a particular case we have a pseudo-Hermitian reduction (**not** a complete description of all admissible reductions).
- We have demonstrated how one can describe an integrable hierarchy of a matrix HF in terms of recursion operators. This is **not** the most general hierarchy related to it however.
- We have applied the Gürses-Karasu-Sokolov method to construct recursion operators and compared them with those obtained in the analysis of recurrence relations of the zero curvature condition.

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Appendix

Let us consider the Lax pair:

$$L(\lambda) = \mathrm{i}\partial_x - \lambda S_1 - \frac{1}{\lambda} S_{-1},$$
$$A(\lambda) = \mathrm{i}\partial_t + \sum_{k=-2,\dots,2} \lambda^k A_k$$

where

$$S_1 = \begin{pmatrix} 0 & \mathbf{u}^T \\ \mathbf{v} & 0 \end{pmatrix}, \qquad S_{-1} = \begin{pmatrix} 0 & K_m \mathbf{u}^T K_n \\ K_n \mathbf{v} K_m & \end{pmatrix}$$

are defined for some $n \times m$ matrices $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t) = \mathcal{E}_n \mathbf{u}^* \mathcal{E}_m$. Moreover, we have

$$\begin{aligned} K_m &= \mathrm{diag}\,(k_1, \dots, k_m), \qquad K_n = \mathrm{diag}\,(k_{m+1}, \dots, k_{m+n}), \qquad k_j^2 = 1, \\ \mathcal{E}_m &= \mathrm{diag}\,(\epsilon_1, \dots, \epsilon_m), \qquad \mathcal{E}_n = \mathrm{diag}\,(\epsilon_{m+1}, \dots, \epsilon_{m+n}), \qquad \epsilon_j^2 = 1 \end{aligned}$$

The above Lax pair is subject to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction

$$\begin{aligned} HL(-\lambda)H &= L(\lambda), \qquad HA(-\lambda)H = A(\lambda), \\ KL(1/\lambda)K &= L(\lambda), \qquad KA(1/\lambda)K = A(\lambda) \end{aligned}$$

where $H = \mathrm{diag}\,(-\mathbb{1}_m, \mathbb{1}_n)$ and $K = \mathrm{diag}\,(K_m, K_n)$. We impose the constraint:

$$\mathbf{u}^T \mathbf{v} \mathbf{u}^T = \mathbf{u}^T.$$

For the condition $[L, A] = 0$ to lead to a local equation it is necessary and sufficient $m = 1$. Then in the pseudo-Hermitian case the equation reads:

$$\mathrm{i}\mathbf{u} + [(\mathbf{u}\mathbf{u}^\dagger \mathcal{E}_n)_x \mathbf{u}]_x + 4(\mathbf{u}^\dagger K_n \mathcal{E}_n \mathbf{u})\mathbf{u} = 0.$$

Constraint:

$$\mathbf{u}^\dagger \mathcal{E}_n \mathbf{u} = 1.$$