# Hierarchies of symplectic structures for $\mathfrak{sl}(3,\mathbb{C})$ Zakharov-Shabat systems in canonical and pole gauge with $\mathbb{Z}_2\times\mathbb{Z}_2$ reduction of Mikhailov type

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# Introduction. Gauge equivalent soliton equations

- The concept of the gauge-equivalent soliton equations originates from the paper of Zakharov and Mikhailov

Zakharov V. and Mikhailov A., *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Soviet Phys. JETP **47** (6) (1978) 1017–1027

- The next example – the gauge equivalence between the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\varphi_t + \varphi_{xx} + 2\varphi|\varphi|^2 = 0, \qquad \lim_{x\to\pm\infty} \varphi(x) = 0$$

and the Heisenberg Ferromagnet equation:

$$S_t = -\frac{1}{2\mathrm{i}}[S, S_{xx}],$$

with $S(x)$ being a $\mathfrak{sl}(2,\mathbb{C})$-valued function such that

$$S^\dagger = -S, \quad \lim_{x\to\pm\infty} S(x) = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^2(x) = \mathbf{1}$$

("†" means Hermitian conjugation). The NLS equation has Lax representation $[L, A] = 0$ where the operator $L$ is defined by the system

$$L\psi = (\mathrm{i}\partial_x + q - \lambda\sigma_3)\psi = 0$$

known as classical Zakharov-Shabat system. Here the 'potential' $q(x)$ is a smooth function of the type

$$\begin{pmatrix} 0 & q_+(x) \\ q_-(x) & 0 \end{pmatrix}, \quad \lim_{x\to\pm\infty} q_\pm(x) = 0.$$

To get the NLS we put $q_+(x) = \varphi(x),\ q_-(x) = \varphi^*(x)$ where $^*$ means complex conjugation. From its side HF equation has a Lax representation $[\tilde{L}, \tilde{A}] = 0$ with $\tilde{L}$

$$(\mathrm{i}\partial_x - \lambda S(x))\tilde{\psi} = \tilde{L}\tilde{\psi} = 0$$

Zakharov and Takhtadjan showed that $L$ and $\tilde{L}$ are gauge equivalent, that is:

$$\tilde{L}=\psi_0^{-1}L\psi_0$$

where $\psi_0$ is the Jost solution of $L$ for $\lambda=0$ having the properties:

$$\lim_{x\to\infty}\psi_0(x)=\mathbf{1},\qquad \lim_{x\to-\infty}\psi_0(x)=T^{-1}(0).$$

($T(\lambda)$ is the transition matrix and in order to ensure the asymptotic of $S$ one must require $T(0)$ to be diagonal).

If NLS has Lax representation $[L,A]=0$ with some $A$ then HF has Lax representation $[\tilde{L},\tilde{A}]=0$ with $\tilde{A}=\psi_0^{-1}A\psi_0$ showing that the gauge-equivalence is a kind of changing variables transformation.

Zakharov V. and Takhtadjan L.*Equivalence between nonlinear Schrödinger equation and Heisenberg ferromagnet equation*, TMF (Theoretical and Mathematical Physics), **38** (1979) 26–35.

The above result has been extended in two directions:

- It has been establish a correspondence between the hierarchies of soliton systems associated with $L$ and $\tilde{L}$, their conservation laws, Hamiltonian structures, etc. This has been achieved through the theory of Recursion Operators $\Lambda$ ( $\tilde{\Lambda}$ ) related to $L$ and $\tilde{L}$ respectively.
- The theory was generalized for the auxiliary linear problems of the type

$$(\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = L\psi = 0$$
$$(\mathrm{i}\partial_x - \lambda S(x))\,\tilde{\psi} = \tilde{L}\tilde{\psi} = 0$$

and the resulting theory is known as gauge-covariant theory of Recursion Operators.

Here $L$ is the so-called generalized Zakharov - Shabat system (GZS) ( when $J$ is real) and Caudrey - Beals - Coifman (CBC) system (when $J$ complex). The potential $q(x)$ and $J$ belong to a fixed simple Lie algebra $\mathfrak{g}$ in some finite dimensional irreducible representation. The element $J$ is regular, that is the kernel of $\mathrm{ad}_J$ is a Cartan subalgebra, $\mathfrak{h}\subset\mathfrak{g}$. $q(x)$ belongs to the orthogonal completion $\mathfrak{h}^\perp=\bar{\mathfrak{g}}$ of $\mathfrak{h}$ with respect to the Killing form:

$$\langle X,Y\rangle=\mathrm{tr}\,(\mathrm{ad}_X\,\mathrm{ad}_Y)\,,\qquad X,Y\in\mathfrak{g},$$

and $\lim_{x\to\pm\infty}q(x)=0$. The system $\tilde{L}$ is referred as GZS (CBC) system in pole gauge in contrast to $L$ which is referred as GZS (CBC) system in canonical gauge. In $\tilde{L}$ the potential $S(x)$ takes values in the orbit of $J$ in the adjoint representation of the group $G$ corresponding to $\mathfrak{g}$ and satisfies $\lim_{x\to\pm\infty}S(x)=J_\pm$. By analogy with $\mathfrak{sl}(2)$ case the first nonlinear systems in the hierarchies of soliton equations corresponding to $L$ and $\tilde{L}$ are called NLS and HF type equations respectively.

- The correspondence between the soliton equations associated with the GZS (CBC) system in pole and canonical gauge goes even further, to the geometric interpretation of the soliton equations and the Recursion Operators. In this interpretation the integrable systems are understood as fundamental fields of a certain Poisson-Nijenhuis structure over the manifold of potentials and the adjoint of $N$ is the Recursion Operator. Then the gauge transformation establishes a map between two P-N manifolds preserving the P-N structure.

Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies – Spectral and Geometric Methods*, Springer, Heidelberg 2008.

The book we cited reflects the situation in the theory prior to 2008. Since then appeared a new trend, namely the challenge to incorporate in the theory the so-called Mikhailov-type reductions. The situation when we have reductions defined by inner isomorphism $h$ of $\mathfrak{g}$ of order $p$ was considered in [Gerdjikov, V. and Yanovski, A. Studies in Appl. Math. (2014)]. The geometric theory has been also studied (for $L$) in [Yanovski A.J. Geom. Symm. Phys. **25** (2012)] and (for $\tilde{L}$) in [Yanovski A. and Vilasi G JNMP, **19** (2012)]. For $\tilde{L}$ in [Yanovski A. and Vilasi, G. SIGMA **8** (2012)]

However, the works treating systems with reductions concentrate either on $L$ or on $\tilde{L}$ and there was little about treating $L$ and $\tilde{L}$ simultaneously.

A comprehensive work that was considering $L$ and $\tilde{L}$ with reductions together appeared only recently [Yanovski A and Valchev T., JNMP **25** (2018)] and involves the so-called GMV system, which we shall introduce now. Since we have here systems on $\mathfrak{sl}(3,\mathbb{C})$ we introduce first the general position case, that is a system without reductions.

# $\mathfrak{sl}(3,\mathbb{C})$ Generating (Recursion) Operators, canonical gauge

We shall take $J = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3)$ ($\lambda$'s are real and distinct). Complex $J$ complicates the spectral theory, but does not change the algebraic properties of the Recursion Operators, besides GMV system corresponds to real $J$. The Cartan subalgebra $\mathfrak{h} = \ker_J$ consists of diagonal matrices and its orthogonal complement $\mathfrak{h}^\perp$ of off-diagonal matrices. We are going to introduce now the soliton equations.

a) Soliton equations related to $L$. Consider the equations having Lax representation $[L, A] = 0$ with $A$ of the form:

$$A = \mathrm{i}\partial_t + \sum_{k=0}^{N} \lambda^k M_k$$

where $M_k$ depend on $q$ and its $x$-derivatives. $[L, A] = 0$ is equivalent to the following system of equations:

$$
\begin{aligned}
&[q, M_0] + \mathrm{i}M_{0;x} - \mathrm{i}q_t = 0\\
&[q, M_1] + \mathrm{i}M_{1;x} - [J, M_0] = 0\\
&[q, M_k] + \mathrm{i}M_{k;x} - [J, M_{k-1}] = 0, \qquad k = 2, 3, \ldots N-1\\
&[q, M_N] + \mathrm{i}M_{N;x} - [J, M_{N-1}] = 0\\
&[J, M_N] = 0
\end{aligned}
$$

It could be shown that it will be enough to take $M_N$ constant, then $M_N \in \mathfrak{h}$. If we denote the orthogonal projector on the space $\mathfrak{h}^\perp$ by $\pi_0$ we shall get immediately that $\pi_0 M_{N-1} = \mathrm{ad}_J^{-1}[q, M_N]$. After that the system is recursively resolved. We are giving the final result, we get that the soliton equation $[L, A] = 0$ is could be written into one of the following equivalent forms:

$$
\begin{aligned}
&\text{a)} \quad \mathrm{i}\,\mathrm{ad}_J^{-1} q_t + \Lambda_\pm^N \left( \mathrm{ad}_J^{-1} [M_N, q] \right) = 0\\
&\text{b)} \quad \mathrm{i}\,\mathrm{ad}_J^{-1} q_t + \Lambda_\pm^{N+1} (M_N) = 0
\end{aligned}
$$

where $\Lambda_\pm$ are the so-called Recursion Operators (called also Generating Operators or $\Lambda$-operators) for GZS in canonical gauge. Their explicit form is

$$\Lambda_\pm(Y(x)) = \mathrm{ad}_J^{-1}\left[\mathrm{i}\partial_x Y + \pi_0(\mathrm{ad}_q Y)(x) + \mathrm{ad}_q \int_{\pm\infty}^{x} ((\mathrm{id} - \pi_0)\mathrm{ad}_q Y)(y)\mathrm{d}y\right]$$

Originally $\Lambda_\pm$ are assumed to act on $Y(x)$ taking values in $\mathfrak{h}^\perp$. However if we make a convention of extending $\mathrm{ad}_J^{-1}$ to act on the whole algebra (putting $\mathrm{ad}_J^{-1}$ equal to zero on the Cartan subalgebra) then it could be extended to act on $Y(x)$ taking values in $\mathfrak{sl}(3,\mathbb{C})$. Also, we could drop $\pi_0$ in the second term of the middle brackets since $\mathrm{ad}_J^{-1}\pi_0 = \mathrm{ad}_J^{-1}$ but the above expression for $\Lambda_\pm$ is familiar from many publications so we prefer to leave it as it is. The coefficients $M_k$, $k < N$ are also calculated through $\Lambda_\pm$.

# $\mathfrak{sl}(3,\mathbb{C})$ Generating (Recursion) Operators, pole gauge

b) Soliton equations related to $\tilde{L}$:

$$\tilde{L}\Psi = (\mathrm{i}\partial_x - \lambda S(x))\Psi = 0, \qquad S(x) \in \mathcal{O}_J$$

where $\mathcal{O}_J$ is the orbit of $J$ (same as in $L$) with respect to the adjoint representation of $\mathrm{SL}(3)$. One sees that $\mathfrak{h}_S = \ker \mathrm{ad}_S$ is a Cartan subalgebra for each $S(x)$ and $S(x)$, $S_1(x) = S^2 - \frac{1}{3}(\mathrm{tr}\, S^2)\mathbf{1}$ span $\mathfrak{h}_S$. Consider now Lax pairs $(\tilde{L}, \tilde{A})$ where $\tilde{A}$'s have the form:

$$\tilde{A} = \mathrm{i}\partial_t + \sum_{k=0}^{N} \lambda^k \tilde{R}_k$$

and the equations having Lax representation $[\tilde{L}, \tilde{A}] = 0$.

It can be shown that if $\tilde{R}_N = aS + bS_1$, $a, b = \mathrm{const}$, then these equations could be written into the form

$$\mathrm{ad}_S^{-1}\partial_t S + \tilde{\Lambda}_\pm^{N-1}(\mathrm{ad}_S^{-1}\tilde{R}_{N,x}) = 0$$

where $\tilde{\Lambda}_\pm$ are the Recursion operators for the system in pole gauge. Its form is complicated, in order to write it we introduce the Gramm matrix $G$ and its inverse $G^{-1}$

$$G = \begin{pmatrix} \langle J, J\rangle & \langle J, J_1\rangle \\ \langle J_1, J\rangle & \langle J_1, J_1\rangle \end{pmatrix} = \begin{pmatrix} 6C_2 & 6C_3 \\ 6C_3 & C_2^2 \end{pmatrix}$$

$$G^{-1} = \frac{1}{12m}\begin{pmatrix} C_2^2 & -6C_3 \\ -6C_3 & 6C_2 \end{pmatrix}$$

where $C_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $C_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3$,
$m = (\lambda_1 - \lambda_2)^2(\lambda_2 - \lambda_3)^2(\lambda_1 - \lambda_3)^2$, $\det(G) = 12m$.

Then

$$\tilde{\Lambda}_\pm(\tilde{Z}) = \mathrm{i}\,\mathrm{ad}_S^{-1}\pi_S\left\{\partial_x\tilde{Z} + (S_x, S_{1;x})G^{-1}\begin{pmatrix} \int\limits_{\pm\infty}^{x}\langle\tilde{Z}, S_y\rangle\mathrm{d}y \\ \int\limits_{\pm\infty}^{x}\langle\tilde{Z}, S_{1;y}\rangle\mathrm{d}y \end{pmatrix}\right\}$$

where $\pi_S$ is the orthogonal projector on the space $\mathfrak{h}_S^\perp$. In case $\tilde{L} = \hat{\psi}_0 L\psi_0$ where $\mathrm{i}\partial_x\psi_0 + q\psi_0 = 0$ we also have

$$\tilde{\Lambda}_\pm = \mathrm{Ad}\,(\psi_0^{-1})\Lambda\pm\mathrm{Ad}\,(\psi_0)$$

# The $GMV_\epsilon$ system

Let us introduce now the GMV system, that is the auxiliary linear problem

$$\tilde{L}_0\psi = (\mathrm{i}\partial_x - \lambda S)\psi = 0, \; S = \begin{pmatrix} 0 & u & v \\ \epsilon u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}, \; \epsilon = \pm 1$$

In the above $(u, v)$ (the potentials) are smooth complex valued functions on $x$ belonging to the real line and $^*$ stands for the complex conjugation. In addition, the functions $u$ and $v$ satisfy the relations:

$$\epsilon|u|^2 + |v|^2 = 1, \quad \lim_{x\to\pm\infty} u(x) = u_\pm, \quad \lim_{x\to\pm\infty} v(x) = v_\pm$$

We call the above system $\mathrm{GMV}_\epsilon$ system. Thus $\mathrm{GMV}_+$ is the usual GMV system which is obtained when one puts $\epsilon = +1$.

The GMV system appears naturally if one considers a system of the type $\mathcal{L}\psi = (\mathrm{i}\partial_x + \lambda R(x))\psi = 0$ where $R(x)$ is $3\times 3$ matrix and soliton equations (nonlinear evolution equations or NLEEs) associated to it, having the form $[\mathcal{L},\mathcal{A}] = 0$ where

$$\mathcal{A}\psi = (\partial_t + \lambda^n \mathcal{A}_n + \ldots + \mathcal{A}_0)\psi = 0$$

and require that we have a Mikhailov reduction group defined by the following action of two generators $g_1, g_2$ on the common fundamental solutions of $\mathcal{L}$ and $\mathcal{A}$:

$$g_1(\psi)(x,\lambda) = \left[Q_\epsilon \psi(x,\lambda^*)^\dagger Q_\epsilon\right]^{-1}, \;\; Q_\epsilon = \mathrm{diag}\,(1,\epsilon,1), \;\; \epsilon = \pm 1$$
$$h_2(\psi)(x,\lambda) = H\psi(x,-\lambda)H$$

where $H = \mathrm{diag}\,(1,1,-1)$. Having a Mikhailov reduction group means that the set of the common fundamental solutions is invariant under $g_1$ and $g_2$. This immediately gives that we must have

$$HRH=-R,\quad H\mathcal{A}_kH=(-1)^k\mathcal{A}_k$$
$$Q_\epsilon R^\dagger Q_\epsilon=R,\quad Q_\epsilon\mathcal{A}_k^\dagger Q_\epsilon=\mathcal{A}_k$$

Since $g_1^2=g_2^2=\mathrm{id}$ and $g_1g_2=g_2g_1$ we have a $\mathbb{Z}_2\times\mathbb{Z}_2$ reduction group.

The requirements on $R$ force it to be of the form as $S$ in $\tilde{L}_0$.

From the other side the requirement $\epsilon|u|^2+|v|^2=1$ ensures that the eigenvalues of $R$ are $0;\pm 1$, that is $R$ belongs to the orbit of the element $J_0=\mathrm{diag}\,(1,0,-1)$ with respect to the adjoint action of the group $\mathrm{SL}\,(3,\mathbb{C})$. This all suggests that $GMV_\epsilon$ is related to some GZS system in canonical gauge.The paper in which GMV was firstly studied in detail was [Gerdjikov, Mikhailov Valchev, SIGMA2011] (for $\epsilon=+1$) and in the case $\lim_{x\to\pm\infty}u=0$. In it were discussed the spectral properties of GMV, the operators whose product play the role of a Recursion Operator, and were given expansions over the so-called adjoint solutions.

However, the study of the GMV system has been done by its own merit and the relation to a GZS system in canonical gauge was not exploited. If one exploits this relation, one is not only able to recover very easy same results, but to generalize to general asymptotic conditions – constant limits $\lim_{x\to\pm\infty} u$ and $\lim_{x\to\pm\infty} v$. One is also able to study both the cases $\epsilon = \pm 1$ simultaneously. Finally, our point of view on the Recursion Operators when reductions are present is also somewhat different from that adopted in the cited work. We presented our results in [Yanovski A and Valchev T., JNMP **25** (2018)]

In fact, in that work we proved the following:

# $\mathrm{GMV}_\pm$ and its gauge-equivalent

## Theorem

*The $\mathrm{GMV}_\pm$ system is gauge equivalent to a canonical GZS linear problem on $\mathfrak{sl}(3,\mathbb{C})$*

$$L_0\psi = (\mathrm{i}\partial_x + q - \lambda J_0)\psi = 0$$

*subject to a Mikhailov reduction group generated by the two elements $h_1$ and $h_2$. On the fundamental solutions $\psi$ of the system $L_0$ we have:*

$$h_1(\psi)(x,\lambda) = \left[Q_\epsilon\psi(x,\lambda^*)^\dagger Q_\epsilon\right]^{-1},\ \ Q_\epsilon = \mathrm{diag}\,(1,\epsilon,1),\ \ \epsilon=\pm1$$
$$h_2(\psi)(x,\lambda) = K\psi(x,-\lambda)K$$

*Since $h_1^2 = h_2^2 = \mathrm{id}$ and $h_1h_2 = h_2h_1$ we have again a $\mathbb{Z}_2\times\mathbb{Z}_2$ reduction.*

In the above

$$J_0 = \mathrm{diag}\,(1, 0, -1), \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Indeed, if $q = \mathrm{i}\psi_0^{-1}(\psi_0)_x$ where

$$\psi_0 = exp[-\mathrm{i}J' \int_{-\infty}^{x} b(y)\mathrm{d}y]g^{-1}$$

$J' = \mathrm{diag}\,(1, -2, 1)$, $b(x) = \frac{\mathrm{i}}{2}(\epsilon u u_x^* + v v_x^*)$ (note that this expression is real) and

$$g = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ \epsilon u^* & \sqrt{2}v & \epsilon u^* \\ v^* & -\sqrt{2}u & v^* \end{pmatrix}, \quad J_0 = \mathrm{diag}\,(1, 0-1)$$

then $\psi_0$ is a solution to $L_0$ for $\lambda = 0$ and $\mathrm{GMV}_\epsilon$ is gauge-equivalent to $L_0$.

# Some algebraic facts

In order to continue some algebraic facts and notation are needed. The two reductions we considered in fact have clear algebraic meaning: $h(X) = HXH$, $k(X) = KXK$ are obviously involutive automorphisms of the algebra $\mathfrak{sl}(3,\mathbb{C})$, and $\sigma_\epsilon X = -Q_\epsilon X^\dagger Q_\epsilon$ defines a complex conjugation of the same algebra.The complex conjugation $\sigma_\epsilon$ defines the real form $\mathfrak{su}(3)$ ($\epsilon = +1$) or the real form $\mathfrak{su}(2,1)$ ($\epsilon = -1$) of $\mathfrak{sl}(3,\mathbb{C})$. In order to treat both cases simultaneously we shall adopt the notation $\mathfrak{su}(\epsilon)$ meaning $\mathfrak{su}(3)$ when $\epsilon = +1$ and $\mathfrak{su}(2,1)$ when $\epsilon = -1$. Let us introduce the spaces

$$\tilde{\mathfrak{g}}^{[j]} = \{X : h(X) = (-1)^j X\}, \quad j = 0,1$$

Then we shall have the orthogonal splitting

$$\mathfrak{sl}(3,\mathbb{C}) = \tilde{\mathfrak{g}}^{[0]} \oplus \tilde{\mathfrak{g}}^{[1]}$$

On the vectors $X$ of the first space $h(X) = X$ (we shall call such a vector ‘positive’ and if $X$ belongs to the second space $h(X) = -X$ (we shall call such a vector ‘negative’.

In a similar way, $\sigma_\epsilon$ splits the algebra $\mathfrak{sl}(3,\mathbb{C})$ into

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{su}(\epsilon)\oplus \mathrm{i}\mathfrak{su}(\epsilon)$$

we shall say that the vectors from the first space are ‘real’, and from the second ‘imaginary’. The invariance under the group generated by $g_1, g_2$ means that if $\psi$ is the common $G_0$-invariant fundamental solution of the linear problem $\tilde{L}\tilde{\psi} = (\mathrm{i}\partial_x - \lambda S(x))\,\tilde{\psi} = 0$ and the linear problem of the type:

$$\tilde{A}\tilde{\psi} = \mathrm{i}\partial_t\tilde{\psi} + (\sum_{k=0}^{n}\lambda^k\tilde{A}_k)\tilde{\psi} = 0, \quad \tilde{A}_k\in\mathfrak{sl}(3,\mathbb{C})$$

we must have $S\in\tilde{\mathfrak{g}}^{[1]}\cap\mathrm{i}\mathfrak{su}(\epsilon)$ and

$$\tilde{A}_{2k+1}\in\mathfrak{g}^{[1]}\cap\mathrm{i}\mathfrak{su}(\epsilon), \quad \tilde{A}_{2k}\in\tilde{\mathfrak{g}}^{[0]}\cap\mathrm{i}\mathfrak{su}(\epsilon), k = 0,1,2,\ldots.$$

Thus $S$ is negative and imaginary. Also, as a direct consequence from the last relations we obtain that

$$h\circ \mathrm{ad}_S = -\mathrm{ad}_S\circ h,\quad \sigma_\epsilon\circ\mathrm{ad}_S = -\mathrm{ad}_S\circ\sigma_\epsilon$$

Consequently, the spaces $\ker\mathrm{ad}_S = \mathfrak{h}_S$ and its orthogonal complement $\mathfrak{h}_S^\perp$ are invariant under $h$ and $\sigma_\epsilon$. Thus these spaces also split into positive-negative and real-imaginary vector subspaces, for example:

$$\mathfrak{h}_S^\perp = \tilde{\mathfrak{f}}^{[0]}\oplus\tilde{\mathfrak{f}}^{[1]},\quad \mathfrak{h}_S = \tilde{\mathfrak{h}}_S^{[0]}\oplus\tilde{\mathfrak{h}}_S^{[1]}$$

Note that the operators $\mathrm{ad}_S$ and $\mathrm{ad}_S^{-1}$ turn positive vectors into negative and vice-versa, that is interchange $\tilde{\mathfrak{f}}^{[0]}$ and $\tilde{\mathfrak{f}}^{[0]}$. Also, they turn real vectors into imaginary and imaginary into real.
Up to now we have seen only one of the facets of the Recursion Operators (called also Generating Operators or Λ-Operators), namely to describe the hierarchies of the NLEEs related to the auxiliary linear problems of GZS type.

# $\mathrm{GMV}_\epsilon$-Recursion Operators

However, the REcursion Operators are theoretical tools that permits also, see the monograph book [Gerdjikov, Vilasi, Yanovski, Integrable Hamiltonian Hierarchies 2008]:

- To describe the hierarchies of conservation laws for these NLEEs.
- To describe the hierarchies of compatible Hamiltonian structures of these NLEEs.
- The expansions over their eigenfunctions permit to interpret the Inverse scattering problems as Generalized Fourier Transforms
- The Recursion Operators have important geometric interpretation, the NLEEs could be viewed as fundamental fields of a Poisson-Nijenhuis structtureh on the infinite dimensional manifold of ‘potentials’.

# $\mathrm{GMV}_\epsilon$-Recursion Operators

The Recursion Operators $\tilde{\Lambda}_\pm$ for $\tilde{L}_0$ could be calculated now immediately using the specific form of $J_0$. We have

$$\tilde{\Lambda}_\pm(Z) =$$

$$\mathrm{i\,ad}_S^{-1}\pi_S\left\{\partial_x Z + \frac{S_x}{12}\int_{\pm\infty}^{x}\langle Z, S_y\rangle dy + \frac{S_{1x}}{4}\int_{\pm\infty}^{x}\langle Z, S_{1y}\rangle dy\right\}$$

We remind that $S_1 = S^2 - \frac{2}{3}\mathbf{1}$ and $S_{1x} = (S_1)_x$.
Because of the way $\mathrm{ad}_S^{-1}$ enters in the expressions for $\tilde{\Lambda}_\pm$ these operators also have the property to turn positive vectors into negative and vice-versa, they also turn real vectors into imaginary and imaginary into real. We shall come a little later to the consequences of this to the hierarchies of soliton equations related to the $GMV_\pm$ problem.

# Symplectis structures for the soliton equations hierarchies

For two functions $X$, $Y$ on $\mathbb{R}$ with values in $\mathfrak{sl}(3,\mathbb{C})$ let $\langle\langle\ ,\ \rangle\rangle$ be the bi-linear form

$$\langle\langle X, Y\rangle\rangle = \int_{-\infty}^{+\infty} \langle X(x), Y(x)\rangle$$

It is well known that the soliton NLEEs are Hamiltonian with respect to hierarchy of symplectic structures – for $L$ and for $\tilde{L}$ respectively they are

$$\Omega^{(m)}(\delta_1 q, \delta_2 q) = \langle\langle \delta_1 q, \Lambda^m \mathrm{ad}_J^{-1}\delta_2 q\rangle\rangle, \quad \Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-)$$

$$\tilde{\Omega}^{(m)}(\delta_1 S, \delta_2 S) = \langle\langle \delta_1 S, \tilde{\Lambda}^m \mathrm{ad}_S^{-1}\delta_2 S\rangle\rangle, \quad \tilde{\Lambda} = \frac{1}{2}(\tilde{\Lambda}_+ + \tilde{\Lambda}_-)$$

The fact that these forms are skew-symmetric is ensured by

## Proposition

*The adjoint of* $\Lambda_+$ $(\Lambda_+)$ *with respect to the bilinear form* $\langle\langle\ ,\ \rangle\rangle$ *is* $\mathrm{ad}_J\Lambda_-\mathrm{ad}_J^{-1}$ $(\mathrm{ad}_J\Lambda_+\mathrm{ad}_J^{-1})$*. Naturally, the adjoint of* $\Lambda$ *is* $\mathrm{ad}_J\Lambda\mathrm{ad}_J^{-1}$*. Similarly the adjoint of* $\tilde{\Lambda}_+$ $(\tilde{\Lambda}_+)$ *with respect to the bilinear form* $\langle\langle\ ,\ \rangle\rangle$ *is* $\mathrm{ad}_S\tilde{\Lambda}_-\mathrm{ad}_S^{-1}$ $(\mathrm{ad}_S\tilde{\Lambda}_+\mathrm{ad}_S^{-1})$ *and the adjoint of* $\tilde{\Lambda}$ *is* $\mathrm{ad}_S\tilde{\Lambda}\mathrm{ad}_S^{-1}$*.*

The fact that they are closed and non-degenerate is more deep. It follows either from the geometric theory or could be obtained directly if one expresses these forms in terms of scattering data.

# The map $q \mapsto S[q]$ and its derivative

In fact the logical way to find the relation between the symplectic structures for GZS system in canonical and pole gauge is the following. We must develop the spectral theory of the operators $\tilde{\Lambda}_\pm$. The first step is to establish the completeness of the eigenfunctions of the Recursion Operators, then to obtain expansion of the 'potentials' $q$ and $S$ in $L_0$ and $\tilde{L}_0$ respectively and their variations $\delta q$ and $\delta S$ over these functions (called generalized exponents). Next, we find the relation between $\delta q$ and $\delta S$, that is the Gateau derivative of the map $q \mapsto S[q]$. Finally we use that derivative in order to find the relation between the hierarchies of symplectic structures. However, this will be impossible to do for the time allocated to this speech, so we shall take a shortcut here.

Suppose $q(x)$ be the potential in the GZS linear problem. Our first goal will be to calculate some useful derivatives. To this end we start with $F([q]) = S = \psi_0^{-1}J\psi_0$. We readily get:

$$\delta S = [S, \psi_0^{-1}\delta\psi_0] = \psi_0^{-1}[J, \delta\psi_0\psi_0^{-1}]\psi_0$$

Next we calculate the derivative $\delta\psi_0$. Taking into account that $\mathrm{i}\partial_x\psi_0 + q\psi_0 = 0$ we obtain $\mathrm{i}\partial_x(\delta\psi_0) + \delta q\psi_0 + q\delta\psi_0 = 0$ and after some calculations we obtain that

$$\delta q = -\mathrm{ad}_J\Lambda_\pm\mathrm{ad}_J^{-1}\pi_0(\psi_0\delta S\psi_0^{-1}) + [B_j, q]\langle B^j, (\delta\psi_0\psi_0^{-1})_\pm\rangle$$

where $\{B_i\}_{i=1}^r$ and $\{B^i\}_{i=1}^r$ are two bi-orthogonal bases of $\mathfrak{h}$ (that is two bases such that $\langle B_i, B^j\rangle = \delta_i^j$), summation over repeated upper and lower index is assumed and $(\delta\psi_0\psi_0^{-1})_\pm = \delta\psi_0\psi_0^{-1}(\pm\infty) = \lim_{x\to\pm\infty}\delta\psi_0\psi_0^{-1}$.

Applying to the both sides $\mathrm{Ad}\,(\psi_0^{-1})$ we get

$$\widetilde{\delta q} = -\mathrm{ad}\,_S\tilde{\Lambda}_{\pm}\mathrm{ad}\,_S^{-1}\delta S + [\widetilde{B}_j,\tilde{q}]\langle B^j,(\delta\psi_0\psi_0^{-1})_{\pm}\rangle$$

where if $Z(x)$ is a function with values in $\mathfrak{g}$ we denote by $\tilde{Z}(x)$ the function $\psi_0^{-1}Z\psi_0$. Thus we finally get

$$\widetilde{\delta q} = -\mathrm{ad}\,_S\tilde{\Lambda}_{\pm}\mathrm{ad}\,_S^{-1}\delta S + \mathrm{i}[\mathrm{ad}\,_S^{-1}S_x,\widetilde{B}_j]\langle B^j,(\delta\psi_0\psi_0^{-1})_{\pm}\rangle$$

In fact if we denote by $F$ the map taking $q$ to $S$ then the above gives the Gateau derivative of $F^{-1}$. It is not hard to see that if $\psi_0$ is the Jost solution $\Psi_0$ for a GZS system in general position, then this formula coincides with the the important formula

$$\tilde{\Lambda}_{+}(\mathrm{ad}_S^{-1}\delta S) =$$
$$\mathrm{Ad}\,(\Psi_0^{-1})\left(\mathrm{ad}\,_J^{-1}\delta q\right) + \mathrm{i}\,\mathrm{ad}\,_S^{-1}\left[\delta T(0)\hat{T}^{-1}(0),\mathrm{ad}\,_S\delta S_x\right]$$

Here $T(0)$ is the value of the transition matrix at $\lambda=0$. This formula is usually obtained in a natural way via spectral theory of GZS system in pole gauge using some Wronskian type relations. Before proceeding further, let us note that there are two type of physically interesting asymptotic conditions for $S$. The first one is a generalization of the asymptotic conditions that one has for the HF equation – then we required $\lim_{x\to\pm\infty} S=\sigma_3$ leading to the requirement that $T(0)$ is diagonal. If for GZS on $\mathfrak{sl}(3,\mathbb{C})$ we require $\lim_{x\to\pm\infty} S=J$ this will be ensured if $T(0)$ is diagonal, that is $T(0)=\exp C_iH_i$ where $H_i$, $i=1,2$ is a basis of $\mathfrak{h}$ of Cartan generators and we assume summation for repeated $i$. Recalling that if one writes a Gauss decomposition of $T(\lambda)$ then the diagonal part components are integrals of motion for the whole hierarchy of NLEEs, we see that $C_i$ are integrals of motion. We shall call this case HF type case.

What is the relation of the symplectic forms hierarchies in the *HF* type case is in fact known for quite long, the formulae on arbitrary semisimple algebra were described in my PhD thesis, but they were not published and in fact are difficult to find. In case of $\mathfrak{sl}(3,\mathbb{C})$ :

$$F^*\Omega^{(m)} = \tilde{\Omega}^{(m+2)} - \frac{\mathrm{i}}{2}\langle H_j, H_k\rangle \, dd_k^{(m+1)} \wedge dC_j$$

where $d_k$ are some integrals of motion for all the NLEEs (we do not specify them here) and there is summation over repeated indexes. The above formula generalizes the result obtained by Kulish and Reinman for the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and shows that $F^*\Omega^{(m)}$ and $\tilde{\Omega}^{(m+2)}$ are dynamically equivalent.

Another type of physically interesting asymptotic behaviour is to have $\lim_{x\to\pm\infty} S(x) = J_\pm$. That kind of asymptotics are displayed in the GMV system. More specifically, for the $GMV_\epsilon$ system $\lim_{x\to-\infty}(\psi_0) = g_-^{-1} = \mathrm{const}$, so that $\delta\psi_0(-\infty) = 0$ and $(\delta\psi_0\psi_0^{-1})_- = 0$. From the other side $\lim_{x\to+\infty}(\psi_0) = g_+^{-1}\exp(-\mathrm{i}\gamma J')$ where

$$\gamma(S) = \int_{-\infty}^{+\infty} \frac{\mathrm{i}}{2}(\epsilon u u_x^* + v v_x^*)\mathrm{d}x$$

It is known that $\gamma$ is a conservation law, so we have

$$(\delta\psi_0\psi_0^{-1})_+ = -\mathrm{i}J'\delta\gamma = -3\mathrm{i}J_1\delta\gamma$$

Taking into account that

$$\mathrm{i}[\mathrm{ad}_S^{-1}S_x,\tilde{B}_j]\langle B^j,J'\rangle = 3\mathrm{i}[\mathrm{ad}_S^{-1}S_x,\tilde{J}_1] = 3\mathrm{i}[\mathrm{ad}_S^{-1}S_x,S_1] = 3\mathrm{i}\,\mathrm{ad}_S^{-1}[S_x,S_1] = -3\mathrm{i}\,\mathrm{ad}_S^{-1}[S,S_{1;x}] = -3\mathrm{i}S_{1;x}$$

we finally obtain that

$$\widetilde{\delta q} = -\mathrm{ad}_S\tilde{\Lambda}_-\mathrm{ad}_S^{-1}\delta S$$
$$\widetilde{\delta q} = -\mathrm{ad}_S\tilde{\Lambda}_+\mathrm{ad}_S^{-1}\delta S - 3\mathrm{i}S_{1;x}\delta\gamma$$
$$\widetilde{\delta q} = -\mathrm{ad}_S\tilde{\Lambda}\mathrm{ad}_S^{-1}\delta S - \tfrac{3\mathrm{i}}{2}S_{1;x}\delta\gamma$$

Thus in GMV case we have a similar formula as in HF case but this with another additional term

$$F^*\Omega^{(m)} = \tilde{\Omega}^{(m+2)} + \frac{3\mathrm{i}}{2}\beta_{m+1}\wedge d\gamma$$

where $\beta_{m+1}$ is the 1-form $\beta_{m+1}(\tilde{X}) = \langle\langle\tilde{\Lambda}^{m+1}\mathrm{ad}_S^{-1}S_{1;x},\tilde{X}\rangle\rangle$ which is also a differential of an integral of motion.

This shows that $F^*\Omega^{(m)}$ and $\tilde{\Omega}^{(m+2)}$ are dynamically equivalent, something that was expected.

However, the invariance with respect to $h$ creates some new features comparing to the case of general position. Indeed, looking at the expression for $\tilde{\Omega}^{(m)}$, we see that depending on $m$ the form is either 'odd' or 'even' with respect to the action of $h$, that is $h^*\tilde{\Omega}^{(m)}=(-1)^{m+1}\tilde{\Omega}^{(m)}$. This shows that for the $GMV_\epsilon$ system all the forms $\tilde{\Omega}^{(m)}$ are equal to zero if $m$ is even, that is only forms odd $m$ 'survive'. The same is true for the forms $\Omega^{(m)}$.

That property is shared also by the additional term, since as could be checked $h^*\gamma=\gamma$ and $h^*\beta_{m+1}=(-1)^{m+1}\beta_{m+1}$.

We consider the hierarchies for GMV and its gauge equivalent. The hierarchy associated with general $L$ has the form:

$$\mathrm{i}\,\mathrm{ad}_J^{-1} q_t + \Lambda_\pm^n \left(\mathrm{ad}_J^{-1}[B, q]\right) = 0, B \in \mathfrak{h}$$

and the gauge-equivalent hierarchy is

$$\mathrm{ad}_S^{-1} \frac{\partial S}{\partial t} + (\tilde{\Lambda}_\pm)^{n-1} \mathrm{ad}_S^{-1}(\partial_x \tilde{B}) = 0$$

Since there is a reduction defined by $h$ we have here some restrictions on $\tilde{B}$ and $n$. Indeed, since $S$ is ‘negative’ and $\mathrm{ad}_S^{-1}, \tilde{\Lambda}$ interchange the spaces $\tilde{\mathfrak{f}}^{[0]}$ and $\tilde{\mathfrak{f}}^{[1]}$, in the above hierarchy the equations that are compatible with the reduction are those for which either $\tilde{B} \in \tilde{\mathfrak{f}}^{[0]}$ (is proportional to $S_{1;x}$) and $n$ is odd, or $\tilde{B} \in \tilde{\mathfrak{f}}^{[1]}$ (is proportional to $S_x$) and $n$ is even. The reduction defined by $\sigma_\epsilon$ leads to the requirement that the constants of proportionality to $S_x$ and $S_{1;x}$ respectively are real.

# Comments and conclusions

We believe that in this work we have made another step towards the gauge-covariant formulation of the theory of the soliton equations associated with the GZS (CBC) system in canonical and in pole gauge when reductions either of the type $\mathbb{Z}_p$ defined by an automorphism of order $p$ or by a complex conjugation are present. (In the case we considered $p = 2$). In general, save to some degeneracy in the symplectic structures and the hierarchies of NLEEs all the features that were present in the case of general position remain true. It will be interesting to investigate what happens if more complicated reduction groups are involved.

Thank you for your attention!

# Λ-Operators, Spectral Theory 1

The properties of the fundamental analytic solutions (FAS) of the GZS systems play a paramount role in the spectral theory of such systems. In fact from the canonical FAS $\chi^\pm$ in canonical gauge one immediately obtains FAS in the pole gauge $\tilde{\chi}^\pm$ (with same analytic properties). Further, one builds the so-called adjoint solutions (or generalized exponents) for the GZS systems:

- GZS system in canonical gauge: $\mathbf{e}^\pm_\alpha = \pi_0\chi^\pm E_\alpha(\chi^\pm)^{-1}$
- GZS system in pole gauge: $\tilde{\mathbf{e}}^\pm_\alpha = \pi_S\tilde{\chi}^\pm E_\alpha(\tilde{\chi}^\pm)^{-1}$

One sees that $\tilde{\mathbf{e}}^\pm_\alpha = \mathrm{Ad}\,(\hat{\psi}_0)\mathbf{e}^\pm_\alpha$ and then the fact that they are eigenfunctions and the completeness relations for them become immediate from the classical results for the Recursion Operators in canonical gauge. Indeed, first

$$\tilde{\Lambda}_-(\tilde{\mathbf{e}}^+_\alpha(x,\lambda)) = \lambda\tilde{\mathbf{e}}^+_\alpha(x,\lambda),\ \tilde{\Lambda}_-(\tilde{\mathbf{e}}^-_{-\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{e}}^-_{-\alpha}(x,\lambda)$$
$$\tilde{\Lambda}_+(\tilde{\mathbf{e}}^+_{-\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{e}}^+_{-\alpha}(x,\lambda),\ \tilde{\Lambda}_+(\tilde{\mathbf{e}}^-_\alpha(x,\lambda)) = \lambda\tilde{\mathbf{e}}^-_\alpha(x,\lambda)$$

# Λ-Operators, Spectral Theory 2

and the completeness relations could be written into the following useful form:

$$\delta(x-y)\tilde{P}_0 = \mathrm{DSC_p} + \frac{1}{2\pi}\int_{-\infty}^{\infty}[\sum_{\alpha\in\Delta_+}\tilde{\mathbf{e}}^+_\alpha(x,\lambda)\otimes\tilde{\mathbf{e}}^+_{-\alpha}(y,\lambda)-\tilde{\mathbf{e}}^-_{-\alpha}(x,\lambda)\otimes\tilde{\mathbf{e}}^-_\alpha(y,\lambda)]\mathrm{d}\lambda$$

where $\mathrm{DSC_p}$ is the discrete spectrum contribution. The second term is the continuous spectrum contribution which we denote by $\mathrm{CSC_p}$. Also, in the above

$$\tilde{P}_0=\sum_{\alpha\in\Delta}\frac{1}{\alpha(J_0)}(\tilde{E}_\alpha\otimes\tilde{E}_{-\alpha}),\quad \tilde{E}_\alpha=\mathrm{Ad}\,(\hat{\psi}_0)E_\alpha=\hat{\psi}_0E_\alpha\psi_0$$

$$\tilde{\mathbf{e}}^\pm_\alpha=\mathrm{Ad}\,(\hat{\psi}_0)\mathbf{e}^\pm_\alpha$$

# Λ-Operators, Spectral Theory 3

For the discrete spectrum contribution we get

$$\mathrm{DSC_p} = -\mathrm{i}\sum_{\alpha\in\Delta_+}\sum_{k=1}^{N^+}\mathrm{Res}(\tilde{Q}^+_\alpha(x,y,\lambda);\lambda^+_k) - \mathrm{i}\sum_{\alpha\in\Delta_+}\sum_{k=1}^{N^-}\mathrm{Res}(\tilde{Q}^-_{-\alpha}(x,y,\lambda);\lambda^-_k)$$

In the above

$$\tilde{Q}^+_\alpha(x,y,\lambda) = \tilde{\mathbf{e}}^+_\alpha(x,\lambda)\otimes\tilde{\mathbf{e}}^+_{-\alpha}(y,\lambda), \quad \mathrm{Im}(\lambda)>0$$
$$\tilde{Q}^-_{-\alpha}(x,y,\lambda) = \tilde{\mathbf{e}}^-_{-\alpha}(x,\lambda)\otimes\tilde{\mathbf{e}}^-_{\alpha}(y,\lambda), \quad \mathrm{Im}(\lambda)<0$$

# Λ-Operators, Reductions 1

## Reduction defined by $h$

For the FAS we have that $H(\tilde{\chi}^+(x,\lambda)) = \tilde{\chi}^-(x,-\lambda)K$ and consequently for $\beta \in \Delta$ we have

$$h(\tilde{\mathbf{e}}^{\pm}_{\beta}(x,\lambda)) = \tilde{\mathbf{e}}^{\mp}_{\mathcal{K}\beta}(x,-\lambda)$$

Changing the variable $\lambda$ to $-\lambda$, taking into account that $\mathcal{K}$ maps the positive roots into the negative ones and vice versa, we obtain after some algebraic transformations

$$\mathrm{CSC_p} =$$

$$\frac{A_h}{2\pi}\int_{-\infty}^{\infty}[\sum_{\alpha\in\Delta_+}\tilde{\mathbf{e}}^+_{\alpha}(x,\lambda)\otimes\tilde{\mathbf{e}}^+_{-\alpha}(y,\lambda)-\tilde{\mathbf{e}}^-_{-\alpha}(x,\lambda)\otimes\tilde{\mathbf{e}}^-_{\alpha}(y,\lambda)]\mathrm{d}\lambda$$

$$A_h = \frac{1}{2}(\mathrm{id} - h\otimes h)$$

Let us explain what is the significance of the above (for simplicity assume we have only continuous spectrum).

# Λ-Operators, Reductions 2

Assume $\tilde{Z}(x)$ is such $h(\tilde{Z}) = \tilde{Z}$ and let us make a contraction, to the right $\eta = +$ and integrate over $y$ and respectively to the left $\eta = -$ and integrate over $x$. Then taking into account that $h$ is automorphism and the Killing form is invariant under automorphisms we get

$$\tilde{Z}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{\alpha\in\Delta_+}\tilde{\mathbf{s}}^{\eta}_{\alpha}(x,\lambda)\mu^{\eta}_{\alpha} - \tilde{\mathbf{s}}^{-\eta}_{-\alpha}(x,\lambda))\mu^{-\eta}_{\alpha}\right]\mathrm{d}\lambda$$

where

$$\mu^{\eta}_{\alpha} = \langle\langle\tilde{\mathbf{a}}^{\eta}_{-\alpha},[S,\tilde{Z}]\rangle\rangle$$
$$\mu^{-\eta}_{\alpha} = \langle\langle\tilde{\mathbf{a}}^{\eta}_{\alpha},[S,\tilde{Z}]\rangle\rangle$$

$$\tilde{\mathbf{s}}^{\eta}_{\pm\alpha}(x,\lambda) = \frac{1}{2}\left(\tilde{\mathbf{e}}^{\eta}_{\pm\alpha}(x,\lambda) + h(\tilde{\mathbf{e}}^{\eta}_{\pm\alpha}(x,\lambda))\right)$$
$$\tilde{\mathbf{a}}^{\eta}_{\pm\alpha}(x,\lambda) = \frac{1}{2}\left(\tilde{\mathbf{e}}^{\eta}_{\pm\alpha}(x,\lambda) - h(\tilde{\mathbf{e}}^{\eta}_{\pm\alpha}(x,\lambda))\right)$$

# Λ-Operators, Reductions 3

If instead of $h(\tilde{Z}) = \tilde{Z}$ we assume that $h(\tilde{Z}) = -\tilde{Z}$ then in the same manner we shall obtain expansions over the functions $\tilde{\mathbf{a}}^{\eta}_{\alpha}$ and the coefficients are calculated via the functions $\tilde{\mathbf{s}}^{\eta}_{\alpha}$. Since $\frac{1}{2}(\mathrm{id} \pm h)$ are in fact projectors onto the $\pm 1$ eigenspaces of $h$

$$h(\tilde{\mathbf{s}}^{\eta}_{\pm\alpha}(x,\lambda)) = \tilde{\mathbf{s}}^{\eta}_{\pm\alpha}(x,\lambda), \quad h(\tilde{\mathbf{a}}^{\eta}_{\pm\alpha}(x,\lambda)) = -\tilde{\mathbf{a}}^{\eta}_{\pm\alpha}(x,\lambda)$$

Thus in case $h(\tilde{Z}) = \tilde{Z}$ or $h(\tilde{Z}) = -\tilde{Z}$ the expansions could be written in terms of new sets of adjoint solutions, $\tilde{\mathbf{s}}^{\eta}_{\pm\alpha}$ that reflect the symmetry of $\tilde{Z}$. For $\alpha \in \Delta_+$ one obtains that

$$\tilde{\Lambda}_-(\tilde{\mathbf{s}}^{+}_{\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{a}}^{+}_{\alpha}(x,\lambda), \quad \tilde{\Lambda}_-(\tilde{\mathbf{s}}^{-}_{-\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{a}}^{-}_{-\alpha}(x,\lambda)$$
$$\tilde{\Lambda}_-(\tilde{\mathbf{a}}^{+}_{\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{s}}^{+}_{\alpha}(x,\lambda), \quad \tilde{\Lambda}_-(\tilde{\mathbf{a}}^{-}_{-\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{s}}^{-}_{-\alpha}(x,\lambda)$$

# Λ-Operators, Reductions 4

and also

$$\tilde{\Lambda}_+(\tilde{\mathbf{s}}^+_{-\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{a}}^+_{-\alpha}(x,\lambda), \quad \tilde{\Lambda}_+(\tilde{\mathbf{s}}^-_{\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{a}}^-_{\alpha}(x,\lambda)$$
$$\tilde{\Lambda}_+(\tilde{\mathbf{a}}^+_{-\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{s}}^+_{-\alpha}(x,\lambda), \quad \tilde{\Lambda}_+(\tilde{\mathbf{a}}^-_{\alpha}(x,\lambda)) = \lambda\tilde{\mathbf{s}}^-_{\alpha}(x,\lambda)$$

One sees that the functions in the expansions when we have some symmetry with respect to $h$ are eigenfunctions for $\tilde{\Lambda}^2_-$ ($\tilde{\Lambda}^2_+$) with eigenvalue $\lambda^2$.This together with the fact that when recursively finding the coefficients for the Lax pairs one effectively uses $\tilde{\Lambda}^2_+$ has leads to the interpretation that in case we have $\mathbb{Z}_2$ reduction defined by $h$ the role of the Generating Operator is played by $\tilde{\Lambda}^2_\pm$.

# Λ-Operators, Reductions 5

**Note that all this happened because of the new form of the expansions, including the 'multiplier' $\frac{1}{2}(\mathrm{id} + h \otimes h)$.**
The the point is that this 'multiplier' has simple algebraic meaning:

## Theorem

*The operator $A_h = \frac{1}{2}(\mathrm{id} + h \otimes h)$ (acting on $\mathfrak{g} \otimes \mathfrak{g}$ where $\mathfrak{g} = \mathrm{sl}(3,\mathbb{C})$) is a projector onto the space*

$$V = (\tilde{\mathfrak{g}}^{[0]} \otimes (\tilde{\mathfrak{g}}^{[1]}) \oplus ((\tilde{\mathfrak{g}}^{[1]} \otimes (\tilde{\mathfrak{g}}^{[0]})$$

Consequently, when for $B \in V$ one makes a contraction (from the right or from the left) with $[S,X]$ where $X$ is in $\tilde{\mathfrak{g}}^{[s]}$, then $[S,X] \in \tilde{\mathfrak{g}}^{[s+1]}$ and $B.([S,X]) \in \tilde{\mathfrak{g}}^{[s]}$.

# $\Lambda$-Operators, Reductions 6

Let us consider now the discrete spectrum term. If we have reduction defined by $h$ we see that we must have $N^+ = N^-$ and if $\tilde{\mathbf{e}}^+_\alpha(x,\lambda)$ has a pole of some order at $\lambda = \lambda^+_s$ in $\mathbb{C}_+$ then $\tilde{\mathbf{e}}^-_{\mathcal{K}\alpha}(x,\lambda)$ will have the same type of singularity at $-\lambda^+_s$ in $\mathbb{C}_-$. In order to simplify the notation we shall put $\lambda^+_s = \lambda_s$, $\lambda^-_s = -\lambda_s$ and $N^+ = N^- = N$. Of course, in order to make some calculations one needs some assumption on the discrete spectrum. Thus assuming that all the singularities are single poles and skipping a lot of technical details, we are able to prove that

$$\mathrm{DSC_p} = -\sum_{\alpha\in\Delta_+}\sum_{s=1}^{2N}\frac{1}{2}(\mathrm{id} - h\otimes h)\mathrm{iRes}(\tilde{Q}_\alpha(x,y,\lambda);\lambda_s)$$

Before, the 'multiplier' $A_h = \frac{1}{2}(\mathrm{id} - h\otimes h)$ lead to expansions over the symmetrized and ant-symmetrized expressions with respect to the action of $h$.

# Λ-Operators Reductions 7

This happens also for the $\mathrm{DSC_p}$, however there are some differences. To each simple pole $\lambda_0$ say of $\tilde{Q}^+_\beta$ corresponds a 2-dimensional invariant space $\tilde{V}^+_\beta$ in which the matrix of $\tilde{\Lambda}_\pm$ is a 2-dimensional Jordan cell with $\lambda_0$ on the diagonal. If we take

$$\tilde{V}^{[+;s]}_\beta = (\mathrm{id} + (-1)^s h)\tilde{V}^+_\beta$$

then this space is not invariant under $\tilde{\Lambda}_\pm$. However, if one considers the action of $\tilde{\Lambda}^2_\pm$ the above space is invariant and one recovers the 2-dimensional Jordan cell structure with $\lambda_0^2$ on the diagonal. **Thus in the presence of $\mathbb{Z}_2$ reduction defined by $h$ for the discrete spectrum the role of the operators $\tilde{\Lambda}_\pm$ is played again by $\tilde{\Lambda}^2_\pm$ – the 2-the power of these operators, just as it was for the continuous spectrum.**

# Λ-Operators Reductions 8

To make the long story short, we have same effect from the reduction defined by the complex conjugation $\sigma_\epsilon$. Both for the continuous and for the discrete spectrum we obtain

$$\mathrm{CSC_p} = A_{\sigma_\epsilon}\mathrm{CSC_p}$$
$$\mathrm{DSC_p} = A_{\sigma_\epsilon}\mathrm{DSC_p}$$

where

$$A_{\sigma_\epsilon} = \frac{1}{2}(\mathrm{id} - \sigma_\epsilon \otimes \sigma_\epsilon)$$

and if we have the two reductions defined by $h$ and $\sigma_\epsilon$ then

$$\mathrm{CSC_p} = A_h A_{\sigma_\epsilon}\mathrm{CSC_p}$$
$$\mathrm{DSC_p} = A_h A_{\sigma_\epsilon}\mathrm{DSC_p}$$

where

$$A_{\sigma_\epsilon} = \frac{1}{2}(\mathrm{id} - \sigma_\epsilon \otimes \sigma_\epsilon), \quad A_h = \frac{1}{2}(\mathrm{id} - h \otimes h)$$

Note that $A_h$ and $A_{\sigma_\epsilon}$ commute.

# Λ-Operators Reductions 9

Of course, in the case of the complex conjugation $\sigma_\epsilon$ the role of 'symmetric' w.r.t. the action of *h* is taken by 'real' functions with respect to $\sigma_\epsilon$ and the role of 'anti-symmetric' w.r.t. the action of *h* is taken by 'imaginary' functions with respect to $\sigma_\epsilon$, etc., etc.

**However, both in the case of one $\mathbb{Z}_2$ reduction, or in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction the role played previously by the operators $\tilde{\Lambda}_\pm$ is played by $\tilde{\Lambda}^2_\pm$ – the 2-the power of these operators.**