Complete Integrability and Separability in Black-Hole Spacetimes

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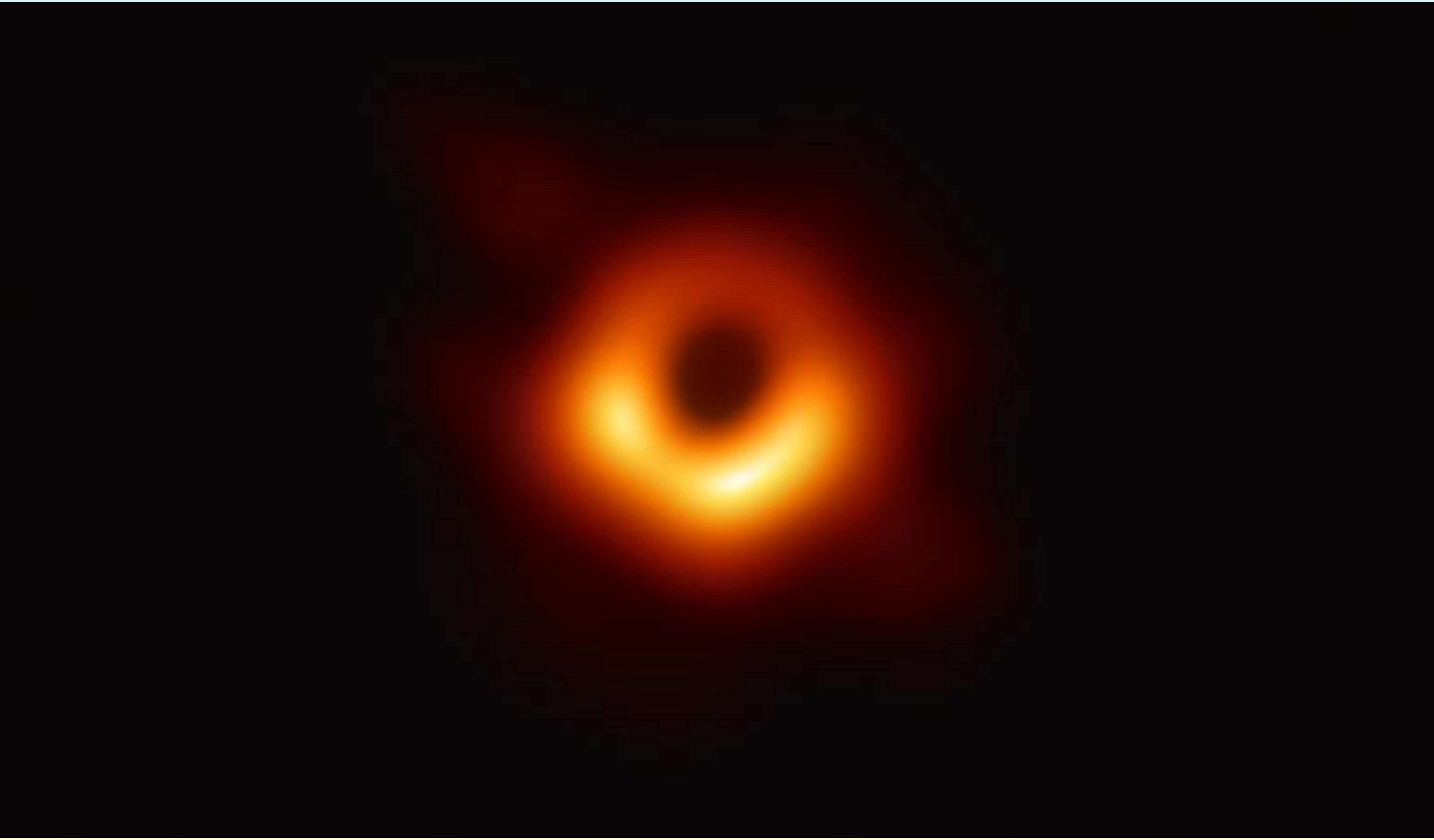
Killam Trust

In total, since making history with the first-ever direct detection of gravitational waves in 2015, the LIGO-Virgo network has spotted evidence for two neutron star mergers, 13 black hole mergers, and one possible black hole-neutron star merger.

“Photo” of a black hole in M87 by EHT.
[6 infrared telescopes form a giant interferometer]



Kerr metric besides evident explicit symmetries possesses also (unexpected) hidden symmetry.

- Complete integrability of geodesic equations [Carter, 1968];
- Complete separability of massless field equations [Carter, 1968; Teukolsky, 1972].

Main problems:

- Higher dimensional generalizations of these results;
- Separability of Proca (massive vector) field equations

“Black Holes, Hidden Symmetry and Complete Integrability”

V.F., Pavel Krtous and David Kubiznak

Living Reviews in Relativity, **20** (2017) no.1, 6; arXiv:1705.05482 (2017)

"Separation of variables in Maxwell equations in Plebanski-Demianski spacetime ",
V. F, P. Krtous and D. Kubiznák, Phys.Rev. D97 (2018) no.10, 101701;

"Separation of Maxwell equations in Kerr-NUT-(A)dS spacetimes" , P. Krtous, V.F.
and D. Kubiznák, Nucl.Phys. B934 (2018) 7-38;

"Massive Vector Fields in Kerr-NUT-(A)dS Spacetimes: Separability and Quasinormal Modes" ,
V.P., P. Krtous, D. Kubiznák, and J. E. Santo, Phys.Rev.Lett. 120 (2018) 231103;

"Duality and μ separability of Maxwell equations in Kerr-NUT-(A)dS spacetimes",
V. F. and P. Krtous, Phys.Rev. D99 (2019) no.4, 044044.

Killing-Yano family

Let ω be p-form on the Riemannian manifold.

$$\nabla_X \omega = \frac{1}{p+1} X \bullet (\nabla \wedge \omega) + \frac{1}{D-p+1} X \wedge (\nabla \bullet \omega) \; [+(\ldots)]$$

If $(\ldots)=0$ ω is a conformal KY tensor.

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If $\nabla \bullet \omega = (\ldots) = 0$, ω is a KY tensor:

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If $\nabla \wedge \omega = (\ldots) = 0$, $\omega = db$ is a CCKY tensor:

$$\nabla_X \omega = \frac{1}{D-p+1} X \wedge (\nabla \bullet \omega)$$

Hodge duality

Let ω be conformal KY p-form

$$\nabla_X \omega = \frac{1}{p+1} X \bullet (\nabla \wedge \omega) + \frac{1}{D-p+1} X \wedge (\nabla \bullet \omega)$$

Then $*\omega$ is also a conformal KY tensor

$$\nabla_X (*\omega) = \frac{1}{p_* + 1} X \bullet (\nabla \wedge *\omega) + \frac{1}{D - p_* + 1} X \wedge (\nabla \bullet *\omega),$$

$$p_* = D - p.$$

Properties of CKY tensor

Hodge dual of CKY tensor is CKY tensor

Hodge dual of CCKY tensor is KY tensor;

Hodge dual of KY tensor is CCKY tensor;

External product of two CCKY tensors is a CCKY tensor

(Krtous,Kubiznak,Page &V.F. '07; V.F. '07)

Killing tensor

A symmetric tensor $k_{ab\ldots c}$ which obeys an equation $\nabla_{(d}k_{ab\ldots c)} = 0$ is called a Killing tensor.

Let $f^{(1)}$ and $f^{(2)}$ are two KY tensors of rank s. Then $k^{ab} = f^{(1)ac_1 \ldots c_{s-1}} f^{(2)b}{}_{c_1 \ldots c_{s-1}}$ is a rank-2 Killing tensor.

Schouten-Nijenhuis bracket

If **a** and **b** are two symmetric tensors of the order r and s, respectively, then $\mathbf{c} = (\mathbf{a}, \mathbf{b})_{SN}$ is again a symmetric tensor of the order $r + s - 1$:

$$c^{a_1 \ldots a_{r-1} c b_1 \ldots b_{s-1}} = r a^{e(a_1 \ldots a_{r-1}} \partial_e b^{c b_1 \ldots b_{s-1})} - s b^{e(b_1 \ldots b_{s-1}} \partial_e a^{c a_1 \ldots a_{r-1})};$$

$\mathbf{c} = (\mathbf{a}, \mathbf{b})_{SN}$ is Schouten – Nijenhuis bracket.

SN bracket is invariant under a change of ∂ to ∇ for any torsion-free corariant derivative.

If $\mathbf{a}$ and $\mathbf{b}$ are two Killing tensors of the rank r and s, respectively, then $\mathbf{c} = (\mathbf{a}, \mathbf{b})_{SN}$ is again a Killing tensor of the rank $r + s - 1$.

Principal tensor

A special case (principal tensor): a non-degenerate closed conformal rank 2 KY tensor h

$$\nabla_X h = \frac{1}{D-1} X \wedge (\nabla \bullet h) \; ;$$

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a , \qquad \xi_a = \tfrac{1}{D\text{-}1} \nabla^b h_{ba} .$$

ξ_a is a (primary) Killing vector. This is easy to show in an Einstein ST, when $R_{ab} = \Lambda g_{ab}$.

Killing-Yano Tower

$CCKY: \ h \Rightarrow \ h^{\wedge j} = \underbrace{h \wedge h \wedge \ldots \wedge h}_{j \text{ times}}$

KY tensors: $k_j = {}^* h^{\wedge j}$

Killing tensors: $K_j = k_j \bullet k_j$

Primary Killing vector: $\xi_{(0)a} = \frac{1}{D-1} \nabla^b h_{ba}$

Secondary Killing vectors: $\xi_{(j)} = K_j \bullet \xi_{(0)}$

A total number of conserved quantities:

$$D = 2n + \varepsilon$$

$$(n+\varepsilon) + (n-1) + 1 = 2n + \varepsilon = D$$

KV *KT* *g*

These Killing vectors and tensors

(i) are independent;

(ii) mutually SN commute.

Relativistic Particle

A relativistic particle in a curved ST:

$$x^a = x^a(\tau), \quad \dot{x}^a \nabla_a \dot{x}^b = 0, \quad \dot{x}^a = \frac{dx^a}{d\tau}.$$

Canonical coordinates in the phase space:

$(x^a, p_a = g_{ab}\dot{x}^b)$. Hamiltonian $H(p,x) = \frac{1}{2} g^{ab} p_a p_b$.

Symplectic form $\Omega = \sum_a dx^a \wedge dp_a$.

The Euler-Lagrange equations $\dot{x}^a \nabla_a \dot{x}^b = 0$ are equivalent to the Poisson equations

$$\dot{x}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial x^a}.$$

Let $k^{a...b}$ be a Killing tensor then $K = k^{a...b} p_a ... p_b$ Poisson-commutes with the Hamiltonian $[K, H] = 0$. Hence, K is an integral of motion.

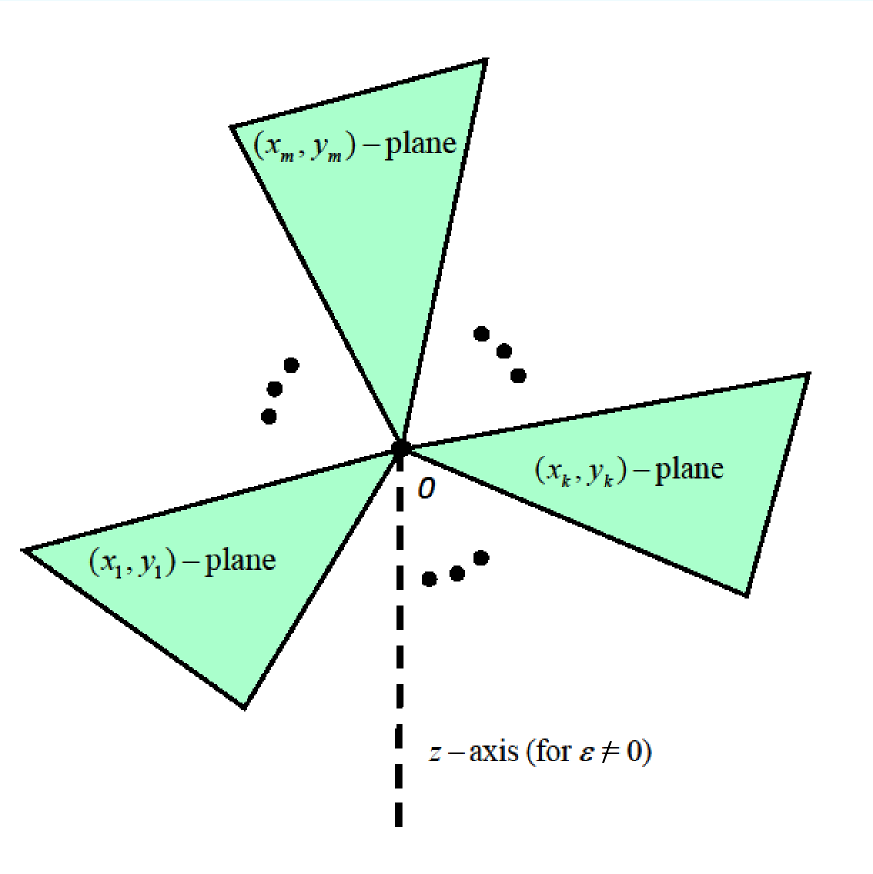
Two integrals of motion are in involution, $[K_1, K_2] = 0$, iff $(k_1, k_2)_{SN} = 0$.

Liouville theorem: Dynamical equations in 2N dimensional phase space are completely integrable if there exist N independent commuting integrals of motion.

This means that a solution of equations of motion of such a system can be obtained by quadratures, i.e. by a finite number of algebraic operations and integrations.

Geodesic equations in a ST admitting the principal tensor are completely integrable.

Canonical coordinates



$$Q_{ab} = h_a{}^c h_{bc}\ ,\ \ \mathrm{Q}\cdot e_\mu = x_\mu^2 e_\mu ,$$

$$h\cdot e_\mu = x_\mu \hat{e}_\mu ,\quad \mathrm{Q}\cdot \hat{e}_\mu = x_\mu^2 \hat{e}_\mu ,$$

Darboux frame:

$$D = 2n + \varepsilon ,\quad \mathrm{h}=\sum_\mu \mathrm{x}_\mu e_\mu \wedge \hat{e}_\mu\ ,$$

$$\mathrm{g}=\sum_\mu (e_\mu e_\mu + \hat{e}_\mu \hat{e}_\mu) + \varepsilon \hat{e}_0 \hat{e}_0$$

Principal tensor *h* is non-denerate: (i) It has *n* different 2-eigenplanes; (ii) x_μ are different and functionally independent in a domain *U* near given point *p*; $\Rightarrow$ (iii) x_μ can be used as *n* coordinates in this domain.

$(n+\varepsilon)$ prime and secondary Killing vectors $\xi_{(j)}$ are commuting. Moreover one has $L_{\xi_{(j)}} h = 0 \;\Rightarrow\; \xi^a_{(j)} x_{\mu,a} = 0.$

According to Frobeneus theorem, there exist local foliation such that $l_{(i)}$ are tangent to *n*-dimensional surfaces $x_\mu = const,$ and one can introduce coordinates $\psi_{\rm i}$ such that $l^a_{(i)}\partial_a = \partial_{\psi_{\rm i}}.$

"General Kerr-NUT-AdS metrics in all dimensions", Chen, Lü and Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$R_{ab} = \frac{2}{D-2}\Lambda g_{ab}, \quad D = 2n + \varepsilon$$

Λ, M – mass, a_k – $(n-1+\varepsilon)$ rotation parameters,

N_α – $(n-1-\varepsilon)$ `NUT' parameters

$$\textit{Total \# of parameters is } 2n-1$$

The metric coefficients of the Kerr-NUT-(A)dS metric written in the canonical coordinates are polynomials of one variable and the parameters of these polynomials are constants that specify a solution. If one substitutes these polynomials by arbitrary functions of the same one variable we call the corresponding metric off-shell.

All Kerr-NUT-AdS metrics and their off-shell extensions in any number of ST dimensions possess a PRINCIPAL TENSOR

(V.F.&Kubiznak '07)

Uniqueness Theorem

A solution of Einstein equations with the cosmological constant, which possesses a PRINCIPAL TENSOR is a Kerr-NUT-AdS metric

(Houri,Oota&Yasui '07 '09;
Krtous, V.F. .&Kubiznak '08;)

The off-shell version of the Kerr-NUT-(A)dS metric possesses the principal tensor.

If the metric possesses a principal tensor it is an off-shell version of the Kerr-NUT-(A)dS metric.

Separability of the Klein–Gordon equation

$$(\Box - m^2)\Phi = 0$$

,

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007)

$$K_0 = \nabla_a (g^{ab} \nabla_b); \quad K_j = -\nabla_a (k_j^{ab} \nabla_b); \quad L_j = -i\, \xi_j^a \nabla_a .$$

$$[K_j, K_k] = [K_j, L_k] = [L_j L_k] = 0;$$

[Sergeev and Krtous (2008); Kolar and Krtous (2015)]

$$K_j \Phi = \kappa_j \Phi, \quad \xi_j \Phi = \lambda_j \Phi;$$

$$\Phi = \prod_{\mu} R_{\mu} \prod_{k=0}^{n-1+\varepsilon} \exp(i\lambda_k \psi_k);$$

$$(X_{\mu} R_{\mu}')' + \varepsilon \frac{X_{\mu}}{x_{\mu}} R_{\mu}' + \frac{Y_{\mu}}{X_{\mu}^2} R_{\mu} = 0.$$

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005 (2007)

Separability of HD Maxwell equations

Maxwell equations in 4D type D ST :

(i) can be decoupled, and (ii) the decoupled equations allow complete separation of variables.

[S.A. Teukolsky, Phys.Rev.Lett., 29, 1114-1118 (1972)]

Method: Newman-Penrose equations;

Special choice of null frames;

$\mathbf{F} = F \pm i * F \quad (anti) - self\text{-}dual$ field

Remarkable progress by Oleg Lunin:

"*Maxwell's equations in the Myers-Perry geometry*", *JHEP* 1712 (2017) 138.

He proposed a special ansatz for em potential in canonical coordinates and demonstrated that Maxwell equations in Kerr-deSitter spacetime are separable.

Our recent results in:

"Separation of variables in Maxwell equations in Plebanski-Demianski spacetime", V. F, P. Krtous and D. Kubiznák, Phys.Rev. D97 (2018) no.10, 101701;

"Separation of Maxwell equations in Kerr-NUT-(A)dS spacetimes", P. Krtous, V.F. and D. Kubiznák, Nucl.Phys. B934 (2018) 7-38.

(i) Covariant description;

(ii) Ansatz for the field in terms of principal tensor **only**;

(iii) Analitical proof of separability;

(iv) Results are valid for any metric which admits the principal tensor $\Rightarrow$ For off-shell Kerr-NUT-(A)dS STs.

Polarization tensor: $\mathbf{B} = \dfrac{1}{1 + i\mu\,\mathbf{h}}$,

$$(g_{ab} + i\mu h_{ab})B^{bc} = \delta_a^c,$$

$$\mathbf{A} = \mathbf{B}\,\nabla Z, \quad Z = \prod_{\mu} R_{\mu} \prod_{k=0}^{n-1+\varepsilon} \exp(i\lambda_k \psi_k).$$

(i) Lorenz condition: $\nabla_a A^a = 0 \Rightarrow$ second order partial differential equation for Z, $DZ = 0$, which allows the separation of variables;

(ii) Maxwell equations $\nabla_b F^{ab} = 0$ in the Lorenz gauge can be written in the form $B^{ab}\nabla_b \tilde{D}Z = 0$. This equation is separable. The separated equations are the same as for the Lorenz equation, with an additional condition that one of sepation constantsis put to zero.

Maxwell equations in 4D

Hodge duality:

$$d\,\mathbf{F} = 0, \quad \mathbf{F} = d\,\mathbf{A}, \quad \delta\mathbf{F} = 0, \quad *\,d\,*\,\mathbf{F} = 0,$$

$$\hat{\mathbf{F}} = *\mathbf{F}, \quad d\,\hat{\mathbf{F}} = 0, \quad \hat{\mathbf{F}} = d\,\hat{\mathbf{A}}, \quad \delta\hat{\mathbf{F}} = 0.$$

$$\mathbf{A} = \mathbf{B}\,\nabla Z, \quad \mathbf{B} = \frac{1}{1 + i\mu\,\mathbf{h}},$$

$$Z = R(r)Y(y)\exp[-i\omega\tau + im\phi].$$

μ-duality

[V. F. and Pavel Krtouš, Phys.Rev. D99 (2019) 044044.]

$$\hat{\mathbf{A}} = \hat{\mathbf{B}}\,\nabla \hat{Z}, \quad \hat{\mathbf{B}} = \frac{1}{1 + i\hat{\mu}\,\mathbf{h}}, \quad \hat{\mu} = -\frac{\omega}{\mu m},$$

$$\hat{Z} = \hat{R}(r)\hat{Y}(y)\exp[-i\omega\tau + im\phi],$$

$$\hat{R} = -\frac{\mu}{\sqrt{-\omega m}}\left(\frac{\Delta_r R'}{q_r} + \frac{\sigma r}{\mu q_r} R\right),$$

$$\hat{Y} = -\frac{\mu}{\sqrt{-\omega m}}\left(\frac{\Delta_y \dot{Y}}{q_y} - \frac{\sigma y}{\mu q_y} Y\right)$$

Massive vector field in Kerr-NUT-(A)dS spacetime

Proca (1936) equation: $\nabla_b F^{ab} + m^2 A^a = 0$

implies Lorenz equation $\nabla_a A^a = 0$

"Constraining the mass of dark photons and axion-like particles through black-hole superradiance",V. Cardoso Ó. Dias,G. Hartnett,M. Middleton,P. Pani,and J.Santos, JCAP 1803 (2018) 043,1475-7516,arXiv:1801.01420

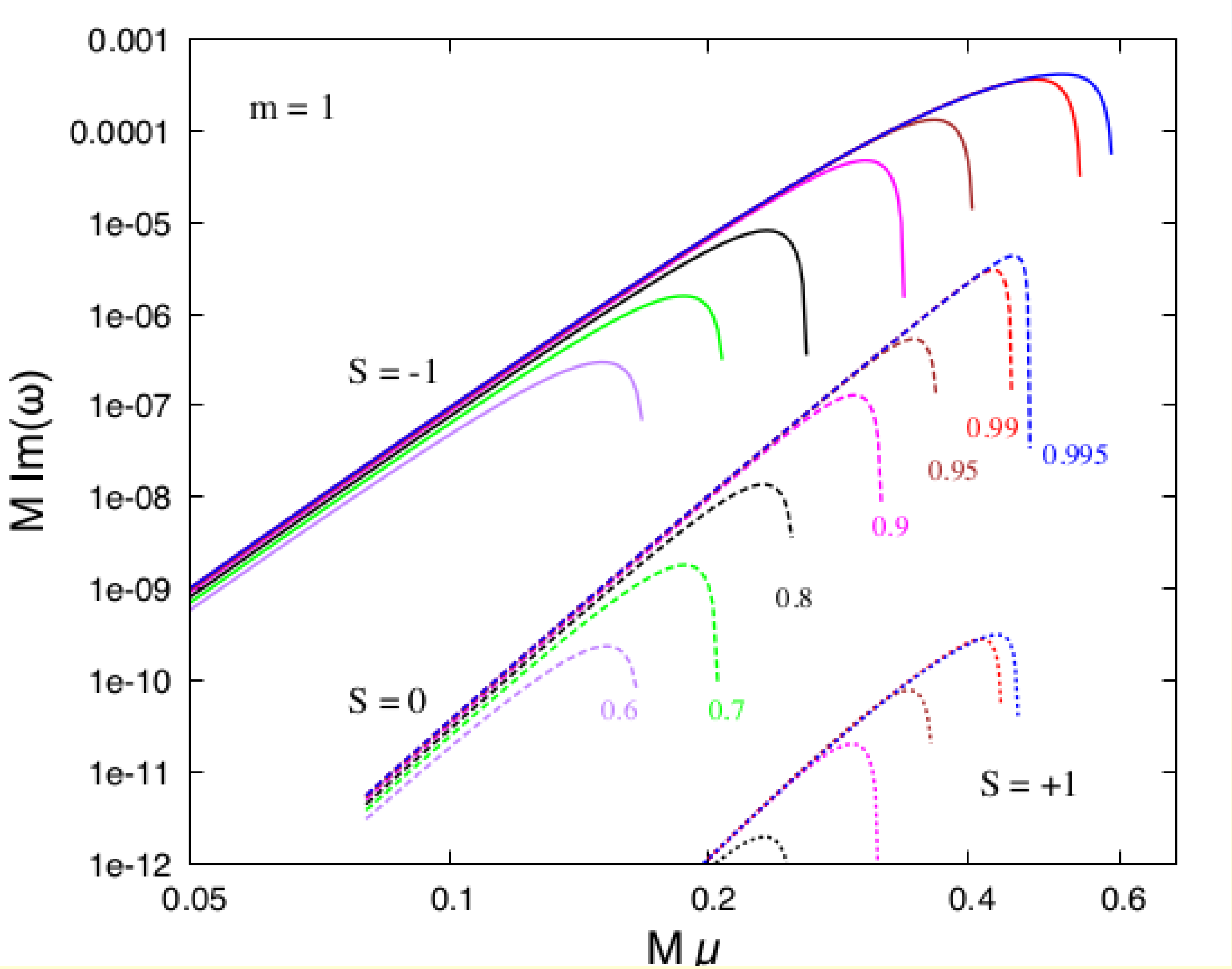
In our recent paper we prove separability of Proca equations in off-shell Kerr-NUT-(A)dS spacetime in any number of dimensions. We use the same ansatz $\mathbf{A} = \mathbf{B}\,\nabla Z$, to solve the Lorenz equation as earlier, and to show that the Proca equation is separable. The only difference is that a separation constant, that for Maxwell eqns vanishes, in the Proca case takes the value $\sim \mathrm{m}^2$.

Ultralight bosons in the dark sector can have potentially observable consequencies for astrophysical black holes.

(i) Gaps in the black hole mass-spin plane, to be revealed by black hole surveys;

(ii) Gravitational wave 'sirens';

(iii) Significant transfers of mass-energy from the black hole into surrounding bosonic 'cloud'.


m = 1
S = -1
S = 0
S = +1
0.6
0.7
0.8
0.9
0.95
0.99
0.995
M Im(ω)
M μ
0.001
0.0001
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
1e-11
1e-12
0.05
0.1
0.2
0.4
0.6

Brief summary

- Spacetimes with a principal tensor possess remarkable properties;
- Complete integrability of geodesic equations;
- Complete separability of important field equations;
- PT allows one to construct canonical coordinates, in which the ST properties are more profound;
- PT $\leftrightarrow$ off-shell vesion of the Kerr-NUT-(A)dS spacetime;
- Arbitrary number of ST dimensions;
- Interesting astrophysical applications (quasi-normal modes of Proca field in the Kerr BH.
- Future work: Separability of HD metric perturbation equations???