Lecture 2: The tangency principle

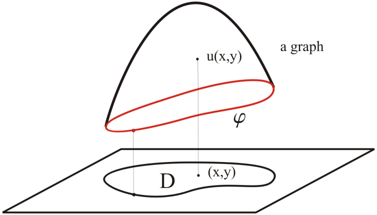
PDEs: a graph $z = u(x, y)$, $(x, y) \in D \subset \mathbb{R}^2$, satisfies

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 2H(1 + u_x^2 + u_y^2)^{3/2}.$$

$$\operatorname{div} \frac{(u_x, u_y)}{\sqrt{1 + u_x^2 + u_y^2}} = \operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 2H.$$

On the boundary

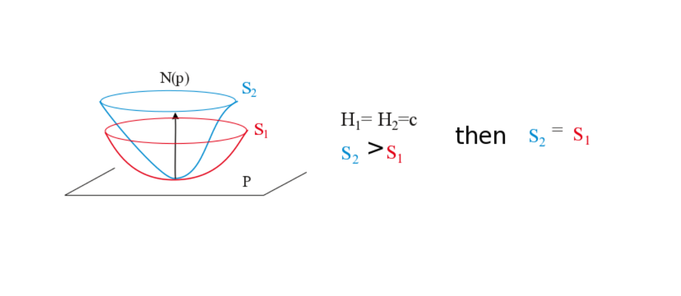
$$u = \varphi, \qquad \text{along } \partial D$$



The difference function $u = u_1 - u_2$ satisfies a linear elliptic PDE:

$$Lu = 0.$$

Theorem (touching-tangency-maximum principle)



Proposition (comparison principle)

If $S_2 \geq S_1$ around p, then $H_2(p) \geq H_1(p)$.

$$(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy} = 2H(1+f_x^2+f_y^2)^{3/2}.$$

After change of coordinates

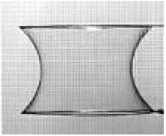
$$2H(p) = \left(\frac{\partial^2 f}{x^2} + \frac{\partial^2 f}{y^2}\right)(p)$$

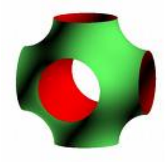
If $f_2 \geq f_1$, $f_2 - f_1$ has local minimum at p, so

$$\frac{\partial f_2}{x^2}(p) \geq \frac{\partial f_1}{x^2}(p), \quad \frac{\partial f_2}{y^2}(p) \geq \frac{\partial f_1}{y^2}(p)$$

There are not **closed compact** *MINIMAL surfaces*

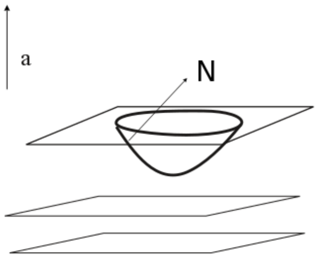
Bounded minimal surfaces with boundary





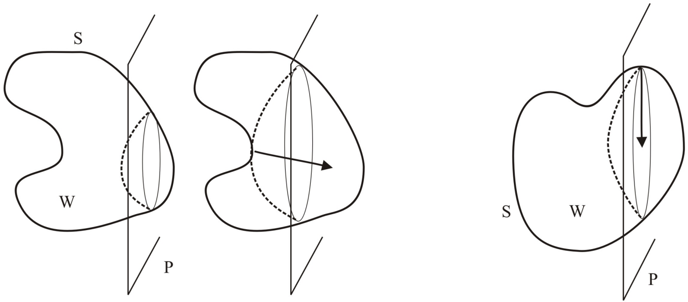
M is H-surface with $\partial M \subset P = \{z = 0\}$

1. If $H = 0$, then M is included in P.
2. For general boundary curve, $H = 0$, then M is included in the convex hull of ∂M.
3. M is a graph, $H > 0$ for $N_3 > 0$. Then $M \subset P^-$.



Theorem (Alexandrov)

Embedded closed *CMC surface* $\Rightarrow$ *round sphere.*

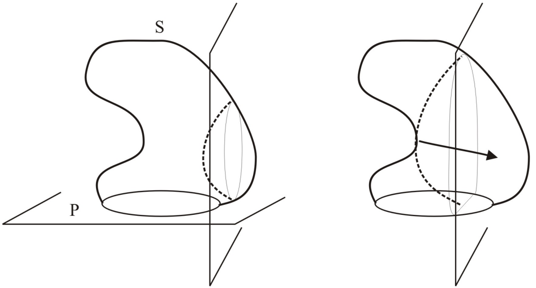


Conjecture 2. *Planar discs and spherical caps are the only compact CMC surfaces with circular boundary that are* **embedded**

Theorem (Alexandrov)

Embedded CMC surface with $\partial M = \mathbb{S}^1$

$$M \subset P^+ \qquad \Rightarrow \quad \text{spherical cap.}$$



Problem. What type of hypothesis ensure that S is over the plane?

Theorem

S CMC embedded surface, $\partial S = C_1 \cup C_2$, C_i coaxial circles in parallel planes. If S lies between P_1 and P_2, then S is rotational.

A liquid drop over a plane is rotational

Theorem (Dirichlet+Neumann)

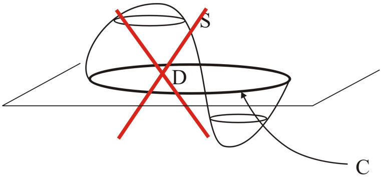
Let S be an embedded CMC surface with $\partial S \subset P$, $S \subset P^+$.
If S makes a constant angle with P along ∂S, then S is a spherical cap.

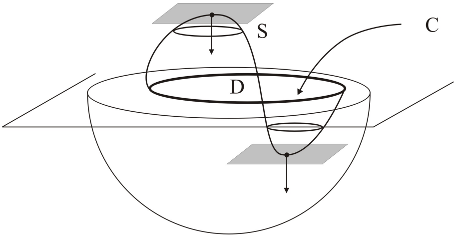
Corollary

A liquid drop between two parallel planes is rotational.

Theorem

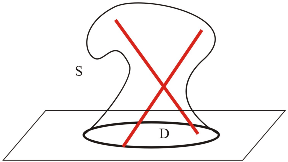
Let S be a CMC embedded surface spanning by C.
If $S \cap ext(D) = \emptyset$, then $S \subset P^+$.

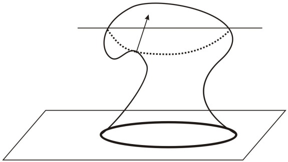




Theorem

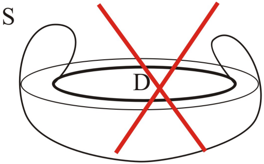
Let S be a CMC embedded and $S \subset P^+$.
If S is a graph around C, then S is a graph.

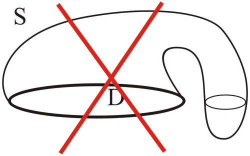




Theorem

Let S be a CMC embedded surface spanning a **convex** *curve. If S is transverse to P along C then $S \subset P^+$.*





Theorem

There do not exist BIG closed liquid drops!!!

liquid BIG drop = liquid drop with weight+embedded surface.
$\rightsquigarrow 2H(x,y,z) = \kappa z + \mu,\ \kappa \neq 0,\ \mu \in \mathbb{R}$.

$$\mathrm{div}\nabla\langle X, \vec{a}\rangle = \Delta\langle X, \vec{a}\rangle = 2H\langle N, \vec{a}\rangle = \kappa z\langle N, \vec{a}\rangle + \mu\langle N, \vec{a}\rangle.$$

$$\kappa\int_S z\langle N, \vec{a}\rangle + \mu\int_S\langle N, \vec{a}\rangle = \int_S \mathrm{div}\nabla\langle X, \vec{a}\rangle = \int_{\partial S=\emptyset} * = 0.$$

$Y = \vec{a} \Rightarrow DIV(Y) = 0 \rightsquigarrow$

$$0 = \int_W DIV(Y) = \int_{\partial W = S}\langle N, Y\rangle = \int_S \langle N, \vec{a}\rangle$$

$Z(x,y,z) = (0,0,z) \rightsquigarrow DIV(Z) = 1.$

$$\mathrm{vol}(W) = \int_W 1 = \int_W DIV(Z) = \int_{\partial W = S}\langle (0,0,z), N\rangle = \int_S z\langle N, \vec{a}\rangle.$$

Stability

Definition

A cmc surface S is stable if

$$A''(0) \geq 0.$$

$$A''(0) = \int_S -f\left(\Delta f + |A|^2 f\right)\, dS \geq 0, \qquad \forall \int_M f \, dS = 0.$$

$$|A|^2 = 4H^2 - 2K, \qquad |A|^2 \geq 2H^2 \quad [(\lambda_1 - \lambda_2)^2 \geq 0]$$

1. If the boundary is fix, we also assume that $f = 0$ along ∂S.
2. If the boundary freely moves in a support, then there is a condition between f and the contact angle.

Theorem

Spheres are the only stable CMC closed surfaces

Proof: find a suitable test function f.

$$\Delta|x|^2 = 4 + 4H\langle N, x\rangle \Rightarrow \int_S 1 + H\langle N, x\rangle = 0$$

$$f = 1 + H\langle N, x\rangle$$

$$\Delta\langle N, x\rangle = -2H - |A|^2\langle N, x\rangle \Rightarrow \Delta f = H(-2H - |A|^2|\langle N, X\rangle)$$

$$\begin{aligned} 0 &\geq \int_S f(\Delta f + |A|^2 f) = \int_S -2H^2(1 + H\langle N, x\rangle) + H|A|^2\langle N, x\rangle + |A|^2 \\ &= \int_S H|A|^2\langle N, x\rangle + |A|^2 \geq \int_S H|A|^2\langle N, x\rangle + 2H^2 \\ &= H\int_S 2H + |A|^2\langle N, x\rangle = 0 \end{aligned}$$

$$\Rightarrow |A| = 2H^2 \rightsquigarrow S \text{ is umbilical (plane and sphere)}$$

Theorem

$\partial M = \mathbb{S}^1$, stable+disc $\Rightarrow$ spherical cap.