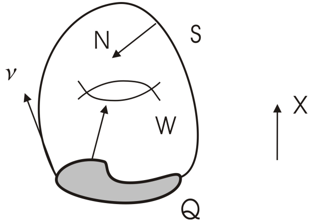
Lecture 3: CMC compact surfaces with boundary

Q1. Does the geometry of the boundary impose restrictions on the geometry of the surface?

> **Question.** *Given a closed curve C and $H \in \mathbb{R}$, is there a H-surface spanning C?*

Example: C is a circle of radius $r > 0$.

Flux formula: $X = a$ is a constant vector field, S and Q two surfaces with $\partial S = \partial Q$ and $W \subset \mathbb{R}^3$ with $\partial W = S \cup Q$.



$$\mathrm{Div}(\vec{a}) = 0 \Rightarrow \int_S \langle N, \vec{a} \rangle + \int_Q \langle N_Q, \vec{a} \rangle = 0.$$

$$\Delta \langle x, \vec{a} \rangle = \mathrm{div}\, (\nabla \langle x, \vec{a} \rangle) = 2H \langle N, \vec{a} \rangle \Rightarrow - \int_{\partial S} \langle \nu, \vec{a} \rangle = \int_S 2H \langle N, \vec{a} \rangle.$$

If H is constant,

$$- \int_{\partial S} \langle \nu, \vec{a} \rangle = 2H \int_S \langle N, \vec{a} \rangle.$$

Theorem (Flux formula I)

$$\int_{\partial S} \langle \nu, \vec{a} \rangle = 2H \int_{Q} \langle N_Q, \vec{a} \rangle$$

$$Z(p) = (p \times \vec{a}) \times N \rightsquigarrow \text{div } Z = -2\langle N, \vec{a} \rangle$$

$$-\int_{\partial S} \langle Z, \nu \rangle = -2 \int_{S} \langle N, \vec{a} \rangle = \frac{1}{H} \int_{\partial S} \langle \nu, \vec{a} \rangle$$

Theorem (Flux formula II)

$$\int_{\partial S} \langle \nu, \vec{a} \rangle = -H \int_{\partial S} \langle \alpha \times \alpha', \vec{a} \rangle.$$

Corollary

If $\partial S = \partial D = C$ *is planar* ($Q = D$, $\vec{a}$ *orthogonal to* D),

$$\int_{\partial S} \langle \nu, \vec{a} \rangle = (\pm) 2H \; Area(D)$$

$$|H| \leq \frac{Length(C)}{2 \; Area(D)}.$$

Corollary

If $C = \mathbb{S}^1(r)$ *then*

$$|H| \leq \frac{1}{r}.$$

Spherical caps show that *all* values of possible H are attained.

- For graphs:

$$div \frac{Du}{\sqrt{1+|Du|^2}} = 2H \Rightarrow \int_D div \frac{Du}{\sqrt{1+|Du|^2}} = \int_D 2H$$

$$2|H|\text{area}(D) = \left| \int_{\partial D} \langle \frac{Du}{\sqrt{1+|Du|^2}} \cdot \mathbf{n} \rangle \right| \leq \int_{\partial D} 1 = L(\partial D).$$

- For constant contact angle: drop in $D \times \mathbb{R}$: $\langle N, \mathbf{n} \rangle = \sin\gamma$:

$$2|H|\text{area}(D) = \cos\gamma L(\partial D)$$

Problem:

Find examples of planar curves where not all values m, $0 < H < \frac{L}{2A}$, exist H-surfaces bounded by C.

$C=$ ellipse? If $a, b > 0$ are the half-axis,

$$|H| \leq \frac{\int_0^{2\pi} \sqrt{a^2 \sin t^2 + b^2 \cos t^2}\ dt}{2\pi ab}.$$

Theorem

If $C = \mathbb{S}^1$ and $|H| = 1$, then S is a hemisphere.

flux formula:

$$2\pi \geq \int_{\partial S} \langle \nu, \vec{a} \rangle = (\pm) 2H \, \text{Area}(D) = \pm 2\pi$$

$\rightsquigarrow \langle \nu, \vec{a} \rangle = 1 \rightsquigarrow \nu = \vec{a}$. Let $\alpha \times \alpha' = \vec{a}$. Thus $H = -1$. Along C:

$$\langle \nu, \vec{a} \rangle^2 + \langle N, \vec{a} \rangle^2 = 1.$$

$$\langle N, \vec{a} \rangle = 0 \Rightarrow \langle N', \vec{a} \rangle = -\sigma_{11} \langle \alpha', \vec{a} \rangle - \sigma_{12} \langle \nu, \vec{a} \rangle = -\sigma_{12}.$$

Then $\sigma_{12} = 0 \rightsquigarrow \alpha$ is a line of curvature.

$$\begin{aligned} \kappa_1 &= \sigma_{11} = -\langle N', \alpha' \rangle = \langle N, \alpha'' \rangle = -\langle N, \alpha \rangle = -\langle \alpha' \times \nu, \alpha \rangle \\ &= -\langle \nu, \vec{a} \rangle = -1 = H. \end{aligned}$$

$$\Rightarrow \kappa_2 = 2H - \kappa_1 = -1$$

$$\kappa_1 = \kappa_2.$$

$\rightsquigarrow$ C s a curve of umbilical points.
On a non-umbilical H-surface, the set of umbilical points

$$\mathcal{U} = \{p \in S : \kappa_1(p) = \kappa_2(p)\}$$

is formed by isolated points [$\Leftarrow \mathcal{U}$ is the zeroes of a holomorphic 2-form]

$\rightsquigarrow$ the surface is umbilical.

Theorem

Let S be a compact minimal surface spanning two planar curves C_1 and C_2 contained in planes P_1 and P_2. If S makes constant angle with P_i along C_i, then P_1 and P_2 are parallel.

If $P_i =< \vec{a}^{\perp} >$, flux formula:

$$\int_C \langle \nu, \vec{a} \rangle = -H \int_C \langle \alpha \times \alpha', \vec{a} \rangle \Rightarrow 0 = \int_{C_1} \nu_1 + \int_{C_2} \nu_2.$$

$$\begin{aligned} \nu_1 &= \langle \nu_1, \vec{a}_1 \rangle \vec{a}_1 + \langle \nu_1, \mathbf{n}_1 \rangle \mathbf{n}_1 \\ &= \cos\gamma_1 \vec{a}_1 + \sin\gamma_1 \mathbf{n}_1. \end{aligned}$$

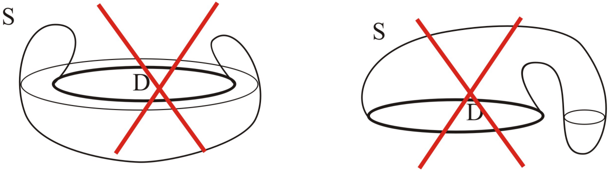
$$\int_{C_1} \nu_1 = \cos\gamma \int_{C_1} \vec{a}_1 + \sin\gamma_1 \int_{C_1} \mathbf{n}_1 = \cos\gamma \int_{C_1} \vec{a}_1.$$

$$\rightsquigarrow 0 = \cos\gamma_1 |C_1| \vec{a}_1 + \cos\gamma_2 |C_2| \vec{a}_2 \Rightarrow \vec{a}_1 || \vec{a}_2.$$

Theorem

CMC embedded surface.

$$C \text{ is convex} + S \text{ transverse to } P \text{ along } C \Rightarrow S \subset P^+.$$



$$-\int_{\partial S}\langle \nu, \vec{a}\rangle = 2H\int_S \langle N, \vec{a}\rangle = -2H\int_D \langle N_D, \vec{a}\rangle.$$

Plateau problem. *Given a closed curve C and $H \in \mathbb{R}$, find a H-surface spanning C with least area and given volume.*

Find a surface $X : \mathbb{D}^2 \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$\Delta X = 2H\, X_u \times X_v$$
$$|X_u|^2 - |X_v|^2 = 0 = \langle X_u, X_v \rangle$$
$$X_{|\partial D} : \partial D \to \Gamma \text{ is a parametrization of } \Gamma$$

Minimize the functional

$$D(X) = \int_D |\nabla X|^2 + \frac{2H}{3} \int_D \langle X_u \times X_v, X \rangle$$

for any immersion X ...

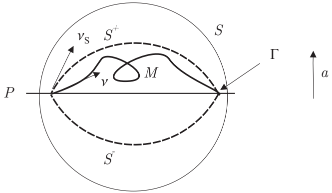
Theorem

If $\Gamma \subset B_{1/|H|}(O)$, there is a solution (with $|X| \le 1/|H|$).

Theorem

If $\partial M = \mathbb{S}^1$ and $M \subset B_{1/H}(O)$, then S is a spherical cap.

If there is a contact point ✓
On the contrary, move B so $B \cap P = \mathbb{S}^1(1/H)$.
Moving up/down, M is included between two small spherical caps



$$|\langle \nu_M, \vec{a} \rangle| < \langle \nu_S, \vec{a} \rangle.$$

$$\left| \int_M \langle \nu_M, \vec{a} \rangle \right| = 2\pi |H| = \left| \int_S \langle \nu_S, \vec{a} \rangle \right|.$$

Conjecture 1 Planar discs and spherical caps are the only compact CMC surfaces with circular boundary that are **topological discs**

Given $C = \mathbb{S}^1$ and $H \in \mathbb{R}$, the area A of a CMC disc with boundary C satisfies $L^2 - 4\pi A + H^2 A^2 \geq 0$.

$$A \leq A_- = \frac{2\pi}{H^2}\left(1 - \sqrt{1 - H^2}\right)$$

or

$$A \geq A_+ = \frac{2\pi}{H^2}\left(1 + \sqrt{1 - H^2}\right)$$

Theorem

Let S be a CMC topological disc, $\partial S = \mathbb{S}^1$. If

$$A \leq A_- = \frac{2\pi}{H^2}\left(1 - \sqrt{1 - H^2}\right),$$

then the surface is a spherical cap.

$$\kappa_n^2 + \kappa_g^2 = 1 \Rightarrow \kappa_g^2 = 1 - \langle N, \alpha \rangle^2 = 1 - \langle \nu, \vec{a} \rangle^2.$$

$$2\pi H = -\int_C \langle \nu, \vec{a} \rangle \Rightarrow (2\pi H)^2 = \left(-\int_C \langle \nu, \vec{a} \rangle\right)^2 \leq 2\pi \int_C \langle \nu, \vec{a} \rangle^2$$

$$\Rightarrow 2\pi H^2 \leq \int \langle \nu, \vec{a} \rangle^2.$$

$$2\pi = \int_S K + \int_{\mathbb{S}^1} \kappa_g \leq AH^2 + \int_C \kappa_g \leq 2\pi - 2\pi\sqrt{1 - H^2} + \int_C \kappa_g.$$

$$4\pi^2(1-H^2) \leq \left(\int_C \kappa_g\right)^2 \leq 2\pi \int_C \kappa_g^2 = 2\pi \int_C (1-\langle \nu, \vec{a}\rangle^2).$$

$$\int_C \langle \nu, \vec{a}\rangle^2 \leq 2\pi H^2$$

$$2\pi H^2 \leq \int_C \langle \nu, \vec{a}\rangle^2 \leq 2\pi H^2.$$