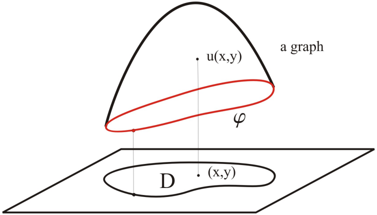
Lecture 4: The Dirichlet problem

Problem: Given a domain $D \subset \mathbb{R}^2$, $H \in \mathbb{R}$ and φ a continuous function on $\partial\Omega$, : Does a graph exist on Ω, with constant mean curvature H and boundary values φ?



$$\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = 2H \text{ on } \Omega$$

$$u = \varphi \text{ along } \partial\Omega$$

Theorem (Serrin)

If Ω is convex with $\kappa(\partial\Omega) > 2H > 0$, YES for any φ.

Assuming $\varphi = 0$:

1. YES for small values of H.
2. If $\partial\Omega$ in convex with $\kappa(\partial\Omega) > H > 0$, YES.
3. If Ω is convex and $\text{area}(\Omega)H^2 < \frac{\pi}{2}$, YES.
4. If Ω is an unbounded convex domain

$$\text{YES} \Leftrightarrow \Omega \subset \text{strip of width } 1/H.$$

For $t \in [0,1]$

$$\begin{cases} Q_t[u] &= (1+|Du|^2)\Delta u - u_i u_j u_{i;j} - t(1+|Du|^2)^{3/2} = 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases}$$

$$\mathcal{A} = \{t \in [0,1] : \exists u_t, Q_t[u_t] = 0, u_{t|\partial\Omega} = 0\}.$$

$1 \in \mathcal{A}$?

- $\mathcal{A} \neq \emptyset$: $0 \in \mathcal{A}$.
- $\mathcal{A}$ is open. If $t_0 \in \mathcal{A}$, $\exists \epsilon > 0$: $(t_0 - \epsilon, t_0 + \epsilon) \subset \mathcal{A}$.
  Define $T(t,u) = Q_t[u]$: $t_0 \in \mathcal{A}$ if and only if $T(t_0, u_{t_0}) = 0$.
  Prove $(DQ_t)_u$ at the point u_{t_0} is an isomorphism, then apply Implicit Function Theorem.
  $\Leftrightarrow$ for any $f \in C^\alpha(\overline{\Omega})$, $\exists_1 v$ $Lv = (DQ_t)_u(v) = f$ in Ω and $v = 0$ on $\partial\Omega$.

$$Lv = (DQ_t)_u v = a_{ij}(Du)v_{i;j} + \mathcal{B}_i(Du, D^2u)v_i,$$

  If is a linear elliptic operator where the standard theory applies.

- $\mathcal{A}$ is closed in $[0,1]$: $\{t_k\} \subset \mathcal{A}$, $t_k \to t \in [0,1]$, $t \in \mathcal{A}$?
  For each $k, \exists u_k : Q_{t_k}[u_k] = 0$ in Ω and $u_k = 0$ in $\partial\Omega$.

$$\mathcal{S} = \{u : \exists t \in [0,1], Q_t[u] = 0, u_{|\partial\Omega} = 0\}.$$

  Then $\{u_k\} \subset \mathcal{S}$. If $\mathcal{S}$ is bounded in 'some' Banach space $-C^{1,\beta}(\overline{\Omega}-$ $\rightsquigarrow$ Schauder theory $\rightsquigarrow$ $\mathcal{S}$ is bounded in $C^{2,\beta}(\overline{\Omega})$ $\rightsquigarrow$ $\mathcal{S}$ is precompact in $C^2(\overline{\Omega})$.
  $\exists \{u_{k_l}\} \subset \{u_k\} \to u \in C^2(\overline{\Omega})$ in $C^2(\overline{\Omega})$. Since $T : [0,1] \times C^2(\overline{\Omega}) \to C^0(\overline{\Omega})$ is continuous,
  $\rightsquigarrow Q_t[u] = T(t,u) = \lim_{l\to\infty} T(t_{k_l}, u_{k_l}) = 0$ in Ω. And

$$u_{|\partial\Omega} = \lim_{l\to\infty} u_{k_l|\partial\Omega} = 0$$

  $\rightsquigarrow u \in C^{2,\alpha}(\overline{\Omega}) \rightsquigarrow t \in \mathcal{A}$.

$\mathcal{A}$ is closed if $\exists M$ independent on $t \in \mathcal{A}$:

$$\|u_t\|_{C^1(\overline{\Omega})} = \sup_{\Omega} |u_t| + \sup_{\Omega} |Du_t| \le M.$$

If $t_1 < t_2$, $t_i \in [0, 1]$, $i = 1, 2$. Then $Q_{t_1}[u_{t_1}] = 0$ and

$$Q_{t_1}[u_{t_2}] = (t_2 - t_1)(1 + |Du_{t_2}|^{3/2}) > 0 = Q_{t_1}[u_{t_1}] = 0.$$

$u_{t_1} = u_{t_2}$ on $\partial\Omega \Rightarrow u_{t_2} < u_{t_1}$ in Ω.

C^0 estimates $\rightsquigarrow$ height estimates
Boundary gradient estimates $\Rightarrow$ Interior gradient estimates
If $u = 0$ on $\partial\Omega$, then boundary gradient estimates $\Leftrightarrow$ estimates of the slope of the graph

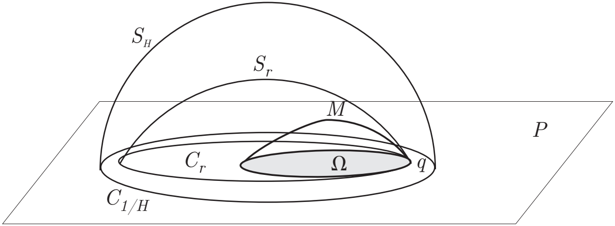
$$1 = \langle N, a\rangle^2 + \langle \nu, a\rangle^2 = \frac{1}{1+|Du|^2} + \langle \nu, a\rangle^2 \Rightarrow \langle \nu, a\rangle = \frac{|Du|}{\sqrt{1+|Du|^2}}.$$

$$\langle \nu, a\rangle \leq C < 1 \Rightarrow |Du| < \frac{C}{\sqrt{1-C^2}}.$$

Theorem

If Ω is a convex domain with $\kappa > H > 0$, then there is a solution of the Dirichlet problem.

Problem: find M such that $|u| < M$, $\langle \nu, a \rangle < C < 1$ along $\partial\Omega$.
Key point: the circle of radius $R = 1/H$ satisfies a rolling condition.



Theorem

If Ω is convex and $L < \sqrt{3}\pi/H^2$, there is a solution with $u = 0$.

$$h \leq \frac{AH}{2\pi} < \frac{1}{2H}.$$