Local Contraction Mapping Principle in Partial Metric Spaces

XXI International Conference on Geometry, Integrability and Quantization

Diana Kirilova Nedelcheva

2019

Banach Fixed Point Theorem 1922

A point x is said to be a fixed point for single - valued mapping $f : X \to X$ if $x = f(x)$.

Let (X, ρ) be a non-empty complete metric space and the mapping $f : X \to X$ is such that there exists $q \in [0, 1)$ such that

$$d(f(x), f(y)) \leq q\rho(x, y) \quad \text{for} \quad \text{all} \quad x, y \in X.$$

Then f admits a unique fixed-point $x^* \in X$ i.e. $f(x^*) = x^*$.

Nadler Fixed Point Theorem 1969

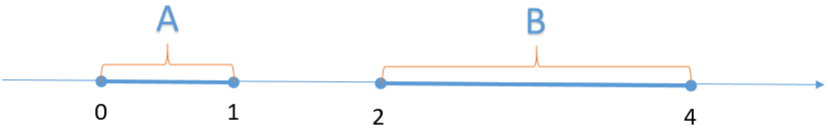
Let $F : X \rightrightarrows X$ is a set - valued mapping i.e. $x \mapsto F(x)$.
A point x is said to be a fixed point for F if $x \in F(x)$

Let (X, ρ) be a non-empty complete metric space. The set - valued mapping $F : X \rightrightarrows X$ is closed valued and there exists $q \in [0, 1)$ such that

$$H(F(x), F(y)) \leq q\rho(x, y) \text{for} \quad \text{all} \quad x, y \in X.$$

Then there exists $x^* \in X$ such that $x^* \in F(x^*)$.

We consider $\mathbb{R}^+$, set $A = [0, 1]$ and set $B = [2, 4]$.



$d(a, b) = |b - a|$.

$d(a, B) = \inf_{b \in B} d(a, b) = |2 - 1| = 1.$

$e(A, B) = \sup_{a \in A} d(a, B) \Rightarrow e(A, B) = d(0, 2) = 2, \; e(B, A) = d(1, 4) = 3.$

$H(A, B) = max\{e(A, B), e(B, A)\} = max\{1, 3\} = 3.$

Dontchev, Hager Fixed Point Theorem 1994

Let (X, ρ) be a non-empty complete metric space. The set - valued mapping $F : X \rightrightarrows X$ is closed valued and there exists $q \in [0, 1)$, $x_0 \in X$, r and q such that $0 \leq q < 1$ and

(i) $d(x_0, F(x_0)) < r(1 - q)$;

(ii) $\mathrm{e}(F(x_1) \cap B_r(x_0), F(x_2)) \leq q\rho(x_1, x_2)$ for all $x_1, x_2 \in B_r(x_0)$.

Then F has a fixed point in $B_r(x_0)$, i.e. there exist $x \in B_r(x_0)$ such that $x \in F(x)$. If F is single - valued, then x is unique.

Partial metric

Let X be a nonempty set. A function $p : X \times X \to R^+$ (where R^+ denotes the set of all nonnegative real numbers) is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

(i) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;

(ii) $p(x,x) \leq p(x,y)$;

(iii) $p(x,y) = p(y,x)$;

(iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space.
The $p-$ ball in X with center $\bar{x}$ and radius r is defined by:

$$\mathbb{B}_r(\bar{x}) = \{x \in X | \ p(x,\bar{x}) < p(\bar{x},\bar{x}) + r\}.$$

Bianchini - Grandolfy function:

Assume that the increasing and continuous functions $\varphi,\ \psi : J \to J$, where J is an interval on $\mathbb{R}_+$ containing 0, are such that:

$(i)\varphi(t) \leq t, \forall t \in J$;

$(ii)\varphi(0) = \psi(0) = 0$;

$(iii)\psi \circ \varphi$ is Bianchini-Grandolfi gauge function such that

$$s(t) = \sum_{n=0}^{\infty} (\psi \circ \varphi)^n(t) < \infty \qquad \text{for} \quad \text{all} \quad t \in J.$$

Main result: Double contraction mapping principle

Let (X, p) and (Y, σ) are complete partial metric spaces and $\bar{x} \in X, \bar{y} \in Y$. Consider the set-valued mappings $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows X$ such that $\bar{x} \in G(\bar{y}), \bar{y} \in F(\bar{x})$. There exists a constant $r > 0$ such that for all $x \in \mathbb{B}_r(\bar{x})$ and for all $y \in \mathbb{B}_r(\bar{y})$, F and G are closed valued.

Suppose that there exists $\alpha \in J \setminus \{0\}$ such that the following assumptions hold:

(a) $d(\bar{x}, G(\bar{y})) < \alpha$;
(b) $d(\bar{y}, F(\bar{x})) < \alpha$, *where* $s(\alpha) \leq \min\{p(\bar{x}, \bar{x}), \sigma(\bar{y}, \bar{y})\} + r$;
(c) $e(F(x_1) \cap \mathbb{B}_r(\bar{y}), F(x_2)) \leq \varphi(p(x_1, x_2))$, *for all* $x_1, x_2 \in \mathbb{B}_r(\bar{x})$;
(d) $e(G(y_1) \cap \mathbb{B}_r(\bar{x}), G(y_2)) \leq \psi(\sigma(y_1, y_2))$, *for all* $y_1, y_2 \in \mathbb{B}_r(\bar{y})$.

Then there exist $x \in \mathbb{B}_r(\bar{x})$ and $y \in \mathbb{B}_r(\bar{y})$ such that $y \in F(x)$ and $x \in G(y)$.

Coupled fixed point

Let $F : X \times X \rightrightarrows X$ be a set-valued mapping.
An element $(x; y) \in X \times X$ is called a coupled fixed point of F if

$$\begin{cases} x \in F(x, y) \\ y \in F(y, x) \end{cases}$$

Corollary: Coupled fixed point theorem

Let (X, p) be a complete partial metric spaces and $\bar{x} \in X$. Consider $G : X \rightrightarrows X$, $F : X \times X \rightrightarrows X$. There exists a constant $r > 0$ such that $F(x, x)$ is nonempty closed subset of X for all $x \in \mathbb{B}_r(\bar{x})$ and $G(x)$ is nonempty closed subset of Y for all $y \in \mathbb{B}_r(\bar{y})$.

Suppose that there exists $\alpha \in J \setminus \{0\}$ such that the following assumptions hold:

(a) $d(\bar{x}, F(\bar{x}, \bar{x})) < \alpha$;
(b) $d(\bar{x}, G(\bar{x})) < \alpha, \quad where \quad s(\alpha) \leq p(\bar{x}, \bar{x}) + r$;
(c) $e(F(x, y) \cap \mathbb{B}_r(\bar{x}), F(u, v)) \leq \varphi(\max\{p(x, u), p(y, v)\})$;
(d) $e(G(x) \cap \mathbb{B}_r(\bar{x}), G(u)) \leq \psi(p(x, u)), \quad for \quad all \quad x, y, u, v \in \mathbb{B}_r(\bar{x})$.

Then there exist $x, y \in \mathbb{B}_r(\bar{x})$ such that $x \in G(F(x, y))$ and $y \in G(F(y, x))$.

THANK YOU FOR YOUR ATTENTION!