FANTASTIC SYMMETRIES AND WHERE TO FIND THEM

Maria Clara Nucci

University of Perugia & INFN-Perugia, Italy

XXI$^{\mathrm{st}}$ International Conference
Geometry, Integrability and Quantization
June 3-8, 2019, Varna, Bulgaria

Lecture 1: Lie and Noether symmetries of differential equations

- Lie symmetry method: a brief review.
- The complete symmetry group.
- Taming chaotic systems by means of Lie symmetry method: an example.
- Lagrangian equations and Noether's first theorem.
- Missed Lie and Noether symmetries: examples.

The role of symmetry in fundamental physics

[David J. Gross, PNAS, 1996]

Symmetry principles play an important role with respect to the laws of nature. They summarize the regularities of the laws that are independent of the specific dynamics. Thus invariance principles provide a structure and coherence to the laws of nature just as the laws of nature provide a structure and coherence to the set of events. Indeed, it is hard to imagine that much progress could have been made in deducing the laws of nature without the existence of certain symmetries.

The role of symmetry in fundamental physics

[David J. Gross, PNAS, 1996]

Symmetry principles play an important role with respect to the laws of nature. They summarize the regularities of the laws that are independent of the specific dynamics. Thus invariance principles provide a structure and coherence to the laws of nature just as the laws of nature provide a structure and coherence to the set of events. Indeed, it is hard to imagine that much progress could have been made in deducing the laws of nature without the existence of certain symmetries.

Along with Frank Wilczek and David Politzer, he was awarded the 2004 Nobel Prize in Physics *for the discovery of asymptotic freedom in the theory of the strong interaction.*

Which symmetries?

"Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry by E. Cartan gained its ascendency in the mathematical community"

Peter J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York (1986)

Which symmetries?

"Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry by E. Cartan gained its ascendency in the mathematical community"

Peter J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York (1986)

More than a quarter of a century later, and after hundred of papers and many books have been published on this subject, the first by Ovsiannikov more than half a century ago, a young theoretical mathematician still states:

Which symmetries?

"Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry by E. Cartan gained its ascendency in the mathematical community"

Peter J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York (1986)

More than a quarter of a century later, and after hundred of papers and many books have been published on this subject, the first by Ovsiannikov more than half a century ago, a young theoretical mathematician still states:

"The only useful symmetry is dilation (they can be used to determine embeddings into Sobelev or Lebesgue spaces) and all other ones are useless"

Which symmetries?

"Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry by E. Cartan gained its ascendency in the mathematical community"

Peter J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, New York (1986)

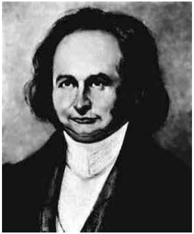
More than a quarter of a century later, and after hundred of papers and many books have been published on this subject, the first by Ovsiannikov more than half a century ago, a young theoretical mathematician still states:

"The only useful symmetry is dilation (they can be used to determine embeddings into Sobelev or Lebesgue spaces) and all other ones are useless"

while a senior theoretical physicist asks:

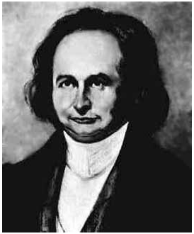
"What are these symmetries?"

Jacobi and Lie: their legacy



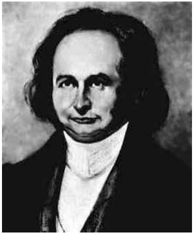
Carl Gustav Jacob Jacobi's (1804-1851) contribution to the study of linear first-order PDEs inspired Sophus Lie (1842-1899) who got convinced that

Jacobi and Lie: their legacy



Carl Gustav Jacob Jacobi's (1804-1851) contribution to the study of linear first-order PDEs inspired Sophus Lie (1842-1899) who got convinced that any integration technique for ordinary differential equations meant the existence of certain groups of symmetries, known as Lie (algebra) groups nowadays.

Jacobi and Lie: their legacy



Carl Gustav Jacob Jacobi's (1804-1851) contribution to the study of linear first-order PDEs inspired Sophus Lie (1842-1899) who got convinced that any integration technique for ordinary differential equations meant the existence of certain groups of symmetries, known as Lie (algebra) groups nowadays. Lie's method has been successfully applied in different problems of physics for more than hundred years, but rarely in biology maybe because the ordinary differential equations studied in this field are generally of first order in contrast with those in physics which are usually of second order.

Ordinary differential equations of first-order

$$\Gamma = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

$$\text{First prolongation}: \underset{1}{\Gamma} = \Gamma + \left(\frac{\mathrm{d}\eta}{\mathrm{d}x} - y'\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)\frac{\partial}{\partial y'}$$

Γ generates a Lie point symmetry for equation:

$$H(x,y,y') \equiv y' - f(x,y) = 0$$

if and only if

$$\underset{1}{\Gamma}\left(H(x,y,y')\right)\Big|_{H=0} = 0$$

This yields the following undetermined determining equation

$$\xi f_x + \eta f_y - \frac{\partial\eta}{\partial x} - \frac{\partial\eta}{\partial y}f + \frac{\partial\xi}{\partial x}f + \frac{\partial\xi}{\partial y}f^2 = 0$$

Ordinary differential equations of first-order

$$\Gamma = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

$$\text{First prolongation} : \underset{1}{\Gamma} = \Gamma + \left(\frac{\mathrm{d}\eta}{\mathrm{d}x} - y'\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)\frac{\partial}{\partial y'}$$

Γ generates a Lie point symmetry for equation:

$$H(x,y,y') \equiv y' - f(x,y) = 0$$

if and only if

$$\underset{1}{\Gamma}\left(H(x,y,y')\right)\Big|_{H=0} = 0$$

This yields the following undetermined determining equation

$$\xi f_x + \eta f_y - \frac{\partial \eta}{\partial x} - \frac{\partial \eta}{\partial y} f + \frac{\partial \xi}{\partial x} f + \frac{\partial \xi}{\partial y} f^2 = 0$$

Link between a Lie symmetry Γ and an integrating factor μ

$$\boxed{\mu = \frac{1}{\eta - f\xi}}$$

EXAMPLE of a first-order ODE

L. Euler, *Novi Commentarii academiae scientiarum Petropolitanae* *17* (1772) 105-124.

$$\frac{dy}{dx} = -nx^{n-2} - \frac{x^{2n-3}}{y}$$

Integrating factor:

$$\mu = (y + x^{n-1})^{-n}$$

Lie symmetry:

$$\Gamma = \frac{\partial}{\partial x} + \left[\frac{(y + x^{n-1})^n}{y} - \frac{nx^{n-2}y + x^{2n-3}}{y} \right] \frac{\partial}{\partial y}$$

A first-order ODE (or system) admits an infinite-dimensional Lie symmetry algebra.
INCREASING THE ORDER DECREASES THE DIMENSION!

Ordinary differential equations of second-order

$$\Gamma = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

Second prolongation:

$$\underset{2}{\Gamma} = \underset{1}{\Gamma} + \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\eta}{\mathrm{d}x} - y'\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) - y''\frac{\mathrm{d}\xi}{\mathrm{d}x}\right]\frac{\partial}{\partial y''}$$

Γ is a Lie point symmetry of the ODE:

$$Eq(x,y,y',y'') \equiv y'' - f(x,y,y') = 0$$

if and only if

$$\underset{2}{\Gamma}\left(Eq(x,y,y',y'')\right)\Big|_{Eq=0} = 0$$

This yields the following determining equation
$\eta_{xy} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})(y')^2 + -(y')^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)f - \xi f_x - \eta f_y - [\eta_x + (\eta_y - \xi_x)y' - (y')^2\xi_y] = 0$
an overdetermined system of linear PDEs.

An EXAMPLE of a second-order ODE

F. Brauer, in *Mathematical Approaches for Emerg. and Reemerg. Inf. Diseases*, Springer, 2002, pp. 31-65.

$$yy'' - y'^2 + y^2y' + yy' + y^3 + y^2 = 0 \quad (\star)$$

An EXAMPLE of a second-order ODE

F. Brauer, in *Mathematical Approaches for Emerg. and Reemerg. Inf. Diseases*, Springer, 2002, pp. 31-65.

$$yy'' - y'^2 + y^2y' + yy' + y^3 + y^2 = 0 \quad (\star)$$

It admits a two-dimensional Lie symmetry algebra generated by:

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = e^x\left(\partial_x - y\partial_y\right)$$

An EXAMPLE of a second-order ODE

F. Brauer, in *Mathematical Approaches for Emerg. and Reemerg. Inf. Diseases*, Springer, 2002, pp. 31-65.

$$yy'' - y'^2 + y^2y' + yy' + y^3 + y^2 = 0 \quad (\star)$$

It admits a two-dimensional Lie symmetry algebra generated by:

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = e^x\,(\partial_x - y\partial_y)$$

Solving

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-y} = \frac{\mathrm{d}y'}{-2y' - y} \Rightarrow X = \log y + x, \quad Y = \frac{y'}{y^2} + \frac{1}{y}$$

An EXAMPLE of a second-order ODE

F. Brauer, in *Mathematical Approaches for Emerg. and Reemerg. Inf. Diseases*, Springer, 2002, pp. 31-65.

$$yy'' - y'^2 + y^2y' + yy' + y^3 + y^2 = 0 \quad (\star)$$

It admits a two-dimensional Lie symmetry algebra generated by:

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = e^x\,(\partial_x - y\partial_y)$$

Solving

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-y} = \frac{\mathrm{d}y'}{-2y' - y} \Rightarrow X = \log y + x, \quad Y = \frac{y'}{y^2} + \frac{1}{y}$$

Then $(\star)$ is reduced to the following first-order autonomous ODE

$$\frac{\mathrm{d}Y}{\mathrm{d}X} + Y + 1 = 0 \Rightarrow (Y+1)\mathrm{e}^X = B \quad \Rightarrow \quad \frac{y'}{y} + 1 + y = B\mathrm{e}^{-x}$$

a standard Riccati equation that yields the gen.sol. of $(\star)$,i.e.

An EXAMPLE of a second-order ODE

F. Brauer, in *Mathematical Approaches for Emerg. and Reemerg. Inf. Diseases*, Springer, 2002, pp. 31-65.

$$yy'' - y'^2 + y^2y' + yy' + y^3 + y^2 = 0 \quad (\star)$$

It admits a two-dimensional Lie symmetry algebra generated by:

$$\Gamma_1 = \partial_x, \qquad \Gamma_2 = e^x\left(\partial_x - y\partial_y\right)$$

Solving

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-y} = \frac{\mathrm{d}y'}{-2y' - y} \Rightarrow X = \log y + x, \quad Y = \frac{y'}{y^2} + \frac{1}{y}$$

Then $(\star)$ is reduced to the following first-order autonomous ODE

$$\frac{\mathrm{d}Y}{\mathrm{d}X} + Y + 1 = 0 \Rightarrow (Y+1)\mathrm{e}^X = B \quad \Rightarrow \quad \frac{y'}{y} + 1 + y = B\mathrm{e}^{-x}$$

a standard Riccati equation that yields the gen.sol. of $(\star)$,i.e.

$$y = \frac{B\mathrm{e}^{-x}}{A\exp\left[B\mathrm{e}^{-x}\right] + 1}$$

Complete symmetry group

In [Krause, J.Math.Phys. 35 (1994)] *an extended notion of symmetry in mechanics* was introduced, to characterize *a classical system by the symmetry laws it obeys*, namely *different mechanical systems cannot have exactly the same symmetry properties.*
Nonlocal symmetries were considered:

$$Y = \left[\int \xi(t, x_1, \ldots, x_N)dt\right] \partial_t + \sum_{k=1}^{N} \eta_k(t, x_1, \ldots, x_N)\partial_{x_k},$$

and then applied to Kepler's problem:

$$Y_1 = 2\left(\int x_1 dt\right) \partial_t + x_1^2\partial_{x_1} + x_1x_2\partial_{x_2} + x_1x_3\partial_{x_3}$$

Complete symmetry group

In [Krause, J.Math.Phys. 35 (1994)] *an extended notion of symmetry in mechanics* was introduced, to characterize *a classical system by the symmetry laws it obeys*, namely *different mechanical systems cannot have exactly the same symmetry properties.*
Nonlocal symmetries were considered:

$$Y = \left[\int \xi(t, x_1, \dots, x_N) dt\right] \partial_t + \sum_{k=1}^{N} \eta_k(t, x_1, \dots, x_N) \partial_{x_k},$$

and then applied to Kepler's problem:

$$Y_1 = 2\left(\int x_1 dt\right) \partial_t + x_1^2 \partial_{x_1} + x_1 x_2 \partial_{x_2} + x_1 x_3 \partial_{x_3}$$

In [MCN, J.Math.Phys.37, 1996] it was shown that they can be retrieved by applying Lie group analysis to the equivalent nonautonomous systems.

Complete symmetry group of the Riccati chain

In [Muriel & Romero, NARWA (2014)] it was shown that the nonlocal complete symmetry group of each member of the Riccati chain can be given by means of λ-symmetries.

Complete symmetry group of the Riccati chain

In [Muriel & Romero, NARWA (2014)] it was shown that the nonlocal complete symmetry group of each member of the Riccati chain can be given by means of λ-symmetries.

In [MCN, TMP 2016] it was shown that indeed those nonlocal symmetries are the complete Lie point symmetry group of the corresponding first-order systems.

Complete symmetry group of the Riccati chain

In [Muriel & Romero, NARWA (2014)] it was shown that the nonlocal complete symmetry group of each member of the Riccati chain can be given by means of λ-symmetries.

In [MCN, TMP 2016] it was shown that indeed those nonlocal symmetries are the complete Lie point symmetry group of the corresponding first-order systems.

Let us begin with the first-order Riccati equation, i.e.

$$\dot{u} = -u^2. \tag{1}$$

Among the infinite number of Lie symmetries, there are the following two:

$$\begin{aligned} \Gamma_1 &= u\left(u\partial_u\right), \\ \Gamma_2 &= u(tu-1)\partial_u, \end{aligned}$$

which form a two-dimensional non-Abelian intransitive Lie algebra (Lie Type IV). It is easy to show that those two symmetries represent the complete symmetry group of equation (1).

Next let us consider the second-order Riccati equation, and write it as a first-order system, i.e.

$$\begin{cases} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= -(3u_1u_2 + u_1^3). \end{cases} \tag{2}$$

Among the infinite number of Lie symmetries, there are the following three:

$$\begin{aligned} \Gamma_1 &= R_1\Big(u_1\partial_{u_1} + (D - u_1)(u_1)\partial_{u_2}\Big), \\ \Gamma_2 &= R_1\Big((tu_1 - 1)\partial_{u_1} + (D - u_1)(tu_1 - 1)\partial_{u_2}\Big), \\ \Gamma_3 &= R_1\Big((t^2u_1 - 2t)\partial_{u_1} + (D - u_1)(t^2u_1 - 2t)\partial_{u_2}\Big), \end{aligned}$$

where $D = \partial_t + u_2\partial_{u_1}$ and $R_1 = u_2 + u_1^2$. Those three symmetries generate the complete symmetry group of system (2).

The third-order equation in the Riccati chain can be written as the following first-order system:

$$\begin{cases} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= u_3, \\ \dot{u}_3 &= -(4u_1u_3 + 3u_2^2 + 6u_1^2u_2 + u_1^4). \end{cases} \tag{3}$$

Among the infinite number of Lie symmetries, there are the following four:

$$\Gamma_1 = R_2\left(u_1\partial_{u_1} + (D - u_1)(u_1)\partial_{u_2} + (D - u_1)^2(u_1)\partial_{u_3}\right),$$

$$\Gamma_2 = R_2\left((tu_1 - 1)\partial_{u_1} + (D - u_1)(tu_1 - 1)\partial_{u_2} + (D - u_1)^2(tu_1 - 1)\partial_{u_3}\right),$$

$$\Gamma_3 = R_2\left((t^2u_1 - 2t)\partial_{u_1} + (D - u_1)(t^2u_1 - 2t)\partial_{u_2} + (D - u_1)^2(t^2u_1 - 2t)\partial_{u_3}\right),$$

$$\Gamma_4 = R_2\left((t^3u_1 - 3t^2)\partial_{u_1} + (D - u_1)(t^3u_1 - 3t^2)\partial_{u_2} + (D - u_1)^2(t^3u_1 - 3t^2)\partial_{u_3}\right),$$

where $D = \partial_t + u_2\partial_{u_1} + u_3\partial_{u_2}$, and $R_2 = u_3 + 3u_1u_2 + u_1^3$. Those four symmetries generate the complete symmetry group of system (3).

Consequently, the complete symmetry group of the first-order system corresponding to the nth equation of the Riccati chain, R_n, is generated by the following $n+1$ operators:

$$\Gamma_1 = R_{n-1}\sum_{j=0}^{n-1}(D-u_1)^j(u_1)\partial_{u_{j+1}},$$

$$\Gamma_1 = R_{n-1}\sum_{j=0}^{n-1}(D-u_1)^j(tu_1-1)\partial_{u_{j+1}},$$

$$\vdots$$

$$\Gamma_{n+1} = R_{n-1}\sum_{j=0}^{n-1}(D-u_1)^j(t^n u_1 - nt^{n-1})\partial_{u_{j+1}},$$

where $D = \partial_t + \sum_{j=1}^{n-1} u_{j+1}\partial_{u_j}$, and R_{n-1} is the $(n-1)$ member of the Riccati chain.

Why not the complete symmetry group

Let us consider again the first-order Riccati equation, i.e.

$$\dot{u} = -u^2.$$

Why not the complete symmetry group

Let us consider again the first-order Riccati equation, i.e.

$$\dot{u} = -u^2.$$

Among the infinite number of Lie symmetries, there are also the following two:

$$X_1 = \partial_t, \qquad X_2 = \left(t - \frac{1}{u}\right)\partial_t,$$

which form another two-dimensional non-Abelian intransitive Lie algebra (Lie Type IV).

Why not the complete symmetry group

Let us consider again the first-order Riccati equation, i.e.

$$\dot{u} = -u^2.$$

Among the infinite number of Lie symmetries, there are also the following two:

$$X_1 = \partial_t, \qquad X_2 = \left(t - \frac{1}{u}\right)\partial_t,$$

which form another two-dimensional non-Abelian intransitive Lie algebra (Lie Type IV).
However, those two symmetries **DO NOT** represent the complete symmetry group since they are also two symmetries of

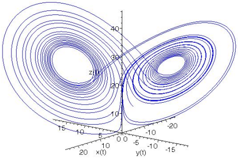
$$\dot{u} = 0.$$

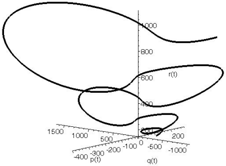
From a butterfly to a tornado with symmetries

In 1972 Edward Norton Lorenz (1917-2008) gave a talk entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?*

From a butterfly to a tornado with symmetries

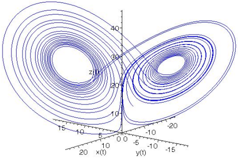
In 1972 Edward Norton Lorenz (1917-2008) gave a talk entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?*

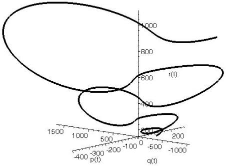




From a butterfly to a tornado with symmetries

In 1972 Edward Norton Lorenz (1917-2008) gave a talk entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?*

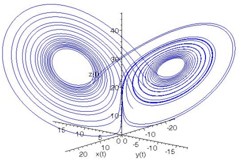


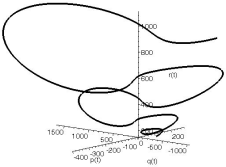


Lie symmetries can transform a butterfly into a tornado...

From a butterfly to a tornado with symmetries

In 1972 Edward Norton Lorenz (1917-2008) gave a talk entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?*





Lie symmetries can transform a butterfly into a tornado...
and a tornado into a butterfly [MCN, J. Math. Phys. 44 (2003)].

Lorenz system (1963)

$$\begin{cases} x' &= \tilde{\sigma}(y - x) \\ y' &= -xz + \tilde{r}x - y \\ z' &= xy - \tilde{b}z \end{cases}$$

Segur's integrable case (1980):

$$\begin{cases} x' &= \frac{(y-x)}{2} \\ y' &= -xz - y \\ z' &= xy - z \end{cases}$$

corresponding third order ODE:

$$2xx''' - 2x'x'' + 5xx'' - 3x'^2 + 2x^3x' + 3xx' + x^4 + x^2 = 0$$

Lie symmetry algebra L_2:

$$X_1 = \partial_\tau, \qquad X_2 = e^{\tau/2}\left(\partial_\tau - \frac{x}{2}\partial_x\right)$$

Euler-Poinsot system (1750)

$$\begin{cases} \dot{p} & = & \frac{(B-C)}{A} qr \\ \dot{q} & = & \frac{(C-A)}{B} pr \\ \dot{r} & = & \frac{(A-B)}{C} pq \end{cases}$$

corresponding third order ODE:

$$p \dddot{p} - \dot{p}\ddot{p} - \frac{4(C-A)(A-B)}{BC} p^3 \dot{p} = 0$$

Lie symmetry algebra $\mathcal{L}_2$:

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = t\partial_t - p\partial_p$$

L_2 and $\mathcal{L}_2$ are the same, i.e., Type IV in Lie's classification.
Transformation from EPS to LIS:

$$\begin{cases} \tau & = & \log(\frac{4}{t^2}) \\ x & = & \frac{p\,t}{2} \\ y & = & \frac{C-B}{2\,A} qrt^2 \\ z & = & \frac{C-B}{2A} \left[\frac{(C-A)}{B} r^2 + \frac{(A-B)}{C} q^2 \right] t^2 \end{cases}$$

and assuming:

$$B = \frac{4A(A-C)}{4A-3C}$$

Transformation from EPS to LIS:

$$\begin{cases} \tau &= \log(\frac{4}{t^2}) \\ x &= \frac{p\,t}{2} \\ y &= -\frac{(2A-C)(2A-3C)}{2A(4A-3C)} qrt^2 \\ z &= -\frac{(2A-C)(2A-3C)\left[4A^2q^2-(4A-3C)^2r^2\right]}{8A^2(4A-3C)^2} t^2 \end{cases}$$

Transformation from LIS to EPS:

$$\begin{cases} t &= 2e^{-\tau/2} \\ p &= xe^{\tau/2} \\ q &= -e^{\tau/2} \frac{(4A-3C)y}{2\sqrt{(2A-C)(2A-3C)(\sqrt{y^2+z^2}+z)}} \\ r &= Ae^{\tau/2} \frac{\sqrt{\sqrt{y^2+z^2}+z}}{\sqrt{(2A-C)(2A-3C)}} \end{cases}$$

and assuming:

$$B = \frac{4A(A-C)}{4A-3C}$$

Transformation from EPS to LIS:

$$\begin{cases} \tau &= \log\left(\frac{4}{t^2}\right) \\ x &= \frac{p\,t}{2} \\ y &= -\frac{(2A-C)(2A-3C)}{2A(4A-3C)}qrt^2 \\ z &= -\frac{(2A-C)(2A-3C)\left[4A^2q^2-(4A-3C)^2r^2\right]}{8A^2(4A-3C)^2}t^2 \end{cases}$$

Transformation from LIS to EPS:

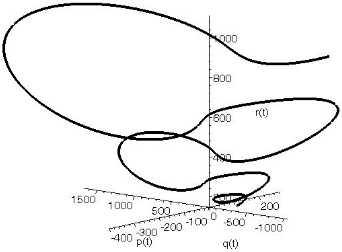
$$\begin{cases} t &= 2e^{-\tau/2} \\ p &= xe^{\tau/2} \\ q &= -e^{\tau/2}\frac{(4A-3C)y}{2\sqrt{(2A-C)(2A-3C)(\sqrt{y^2+z^2}+z)}} \\ r &= Ae^{\tau/2}\frac{\sqrt{\sqrt{y^2+z^2}+z}}{\sqrt{(2A-C)(2A-3C)}} \end{cases}$$

What happens if one applies the above transformation to the general Lorenz system?

$$
\left\{
\begin{array}{rcl}
\dot{p} & = & \frac{2(2A-C)(2A-3C)\tilde{\sigma}}{A(4A-3C)} qr + (2\tilde{\sigma}-1)\,\frac{p}{t} \\
\dot{q} & = & \frac{3C-4A}{4A} pr + (\tilde{b}-1)\frac{4A^2q^2-(4A-3C)^2r^2}{4A^2q^2+(4A-3C)^2r^2}\,\frac{q}{t} \\
 & & +\tilde{r}\frac{2(4A-3C)^3A}{(2A-C)(2A-3C)[4A^2q^2+(4A-3C)^2r^2]}\,\frac{pr}{t^2} \\
\dot{r} & = & \frac{A}{4A-3C} pq - (\tilde{b}-1)\frac{4A^2q^2-(4A-3C)^2r^2}{4A^2q^2+(4A-3C)^2r^2}\,\frac{r}{t} \\
 & & +\tilde{r}\frac{8(4A-3C)A^3}{(2A-C)(2A-3C)[4A^2q^2+(4A-3C)^2r^2]}\,\frac{pq}{t^2}
\end{array}
\right.
$$

$$
\begin{cases}
\dot{p} = \dfrac{2(2A-C)(2A-3C)\tilde{\sigma}}{A(4A-3C)} qr + (2\tilde{\sigma}-1)\,\dfrac{p}{t} \\
\dot{q} = \dfrac{3C-4A}{4A} pr + (\tilde{b}-1)\dfrac{4A^2q^2-(4A-3C)^2r^2}{4A^2q^2+(4A-3C)^2r^2}\,\dfrac{q}{t} \\
\qquad +\tilde{r}\dfrac{2(4A-3C)^3A}{(2A-C)(2A-3C)[4A^2q^2+(4A-3C)^2r^2]}\,\dfrac{pr}{t^2} \\
\dot{r} = \dfrac{A}{4A-3C} pq - (\tilde{b}-1)\dfrac{4A^2q^2-(4A-3C)^2r^2}{4A^2q^2+(4A-3C)^2r^2}\,\dfrac{r}{t} \\
\qquad +\tilde{r}\dfrac{8(4A-3C)A^3}{(2A-C)(2A-3C)[4A^2q^2+(4A-3C)^2r^2]}\,\dfrac{pq}{t^2}
\end{cases}
$$

A tornado appears!



About 101 years ago

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

About 101 years ago

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

For those differential equations that arise from variational problems, the statements that can be formulated are much more precise than for the arbitrary differential equations that are invariant under a group, which are the subject of Lie's researches. What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups.

We refer to the excellent book Kosmann-Schwarzbach, 2011 for the historical background and thorough account of the developments of Noether's work in various fields, and to the book by Olver, 1986, 1993 for a modern mathematical formulation.
As tersely stated in Olver, Forum of Mathematics, Sigma, 2018:
The First Noether Theorem establishes the connection between continuous variational symmetry groups and conservation laws of their associated Euler-Lagrange equations. The Second Noether Theorem deals with the case when the variational symmetry group is infinite-dimensional, depending on one or more arbitrary functions of the independent variables, e.g., the gauge symmetry groups arising in relativity and physical field theories.

We refer to the excellent book Kosmann-Schwarzbach, 2011 for the historical background and thorough account of the developments of Noether's work in various fields, and to the book by Olver, 1986, 1993 for a modern mathematical formulation.
As tersely stated in Olver, Forum of Mathematics, Sigma, 2018: *The First Noether Theorem establishes the connection between continuous variational symmetry groups and conservation laws of their associated Euler-Lagrange equations. The Second Noether Theorem deals with the case when the variational symmetry group is infinite-dimensional, depending on one or more arbitrary functions of the independent variables, e.g., the gauge symmetry groups arising in relativity and physical field theories.*
Here we are mainly concern with Noether's First Theorem.
Therefore, what we call Noether's theorem below is actually Noether's First Theorem.

Lagrangian equations and Noether's first theorem

Variational problem most familiar to physicists:

$$L = L(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)),$$

with its corresponding (Euler)-Lagrangian equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}, \quad (k = 1, \ldots, n).$$

Then a Noether symmetry has to satisfy the following relationship:

$$L\frac{d\xi}{dt} + \underset{1}{\Gamma}(L) = \frac{df}{dt},$$

where $f = f(t, \mathbf{q})$ is a function to be determined, and Γ_1 is the first prolongation of $\Gamma = \xi(t, \mathbf{q})\partial_t + \sum_{k=1}^{n} \eta_k(t, \mathbf{q})\partial_{q_k}$.

Lagrangian equations and Noether's first theorem

Variational problem most familiar to physicists:

$$L = L(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)),$$

with its corresponding (Euler)-Lagrangian equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}, \quad (k = 1, \ldots, n).$$

Then a Noether symmetry has to satisfy the following relationship:

$$L\frac{d\xi}{dt} + \underset{1}{\Gamma}(L) = \frac{df}{dt},$$

where $f = f(t, \mathbf{q})$ is a function to be determined, and Γ_1 is the first prolongation of $\Gamma = \xi(t, \mathbf{q})\partial_t + \sum_{k=1}^{n} \eta_k(t, \mathbf{q})\partial_{q_k}$. Thus, Noether's theorem yields the following first integral:

$$I = \xi L + \sum_{k=1}^{n} \frac{\partial\, L}{\partial \dot{q}_k}(\eta_k - \dot{q}_k\xi) - f = \mathrm{cost}.$$

Missing Noether symmetries

The key to find Noether symmetries is the boundary term f. In Mechanics courses, students are usually taught very simple Noether symmetries of the natural Lagrangian (namely, Kinetic minus Potential energy), e.g., translation on time, that yield $f=$constant. Indeed, one may have to deal with a very complicated f in order to find a Noether symmetry.

Missing Noether symmetries

The key to find Noether symmetries is the boundary term f. In Mechanics courses, students are usually taught very simple Noether symmetries of the natural Lagrangian (namely, Kinetic minus Potential energy), e.g., translation on time, that yield f=constant. Indeed, one may have to deal with a very complicated f in order to find a Noether symmetry.

Fang et al, Phys. Lett. A, 2010 presented the following Lagrangian

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}k(q_1^2 + q_2^2) + tq_1, \quad (k, m = \mathrm{const}), \tag{4}$$

and its corresponding Lagrangian equations

$$\ddot{q}_1 = -\frac{k}{m}q_1 + \frac{t}{m}, \qquad \ddot{q}_2 = -\frac{k}{m}q_2. \tag{5}$$

There three first integrals were determined, with the claim that only one was related to Noether symmetries.

Missing Noether symmetries

The key to find Noether symmetries is the boundary term f. In Mechanics courses, students are usually taught very simple Noether symmetries of the natural Lagrangian (namely, Kinetic minus Potential energy), e.g., translation on time, that yield f=constant. Indeed, one may have to deal with a very complicated f in order to find a Noether symmetry.

Fang et al, Phys. Lett. A, 2010 presented the following Lagrangian

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}k(q_1^2 + q_2^2) + tq_1, \quad (k, m = \mathrm{const}), \tag{4}$$

and its corresponding Lagrangian equations

$$\ddot{q}_1 = -\frac{k}{m}q_1 + \frac{t}{m}, \qquad \ddot{q}_2 = -\frac{k}{m}q_2. \tag{5}$$

There three first integrals were determined, with the claim that only one was related to Noether symmetries.

In *MCN, Phys. Lett. A, 2011*, eight Noether symmetries and corresponding conserved quantities were derived.

System (5) is linear therefore it admits a fifteen-dimensional Lie point symmetry algebra, isomorphic to $sl(4,\boldsymbol{R})$. Then, the Lagrangian (4) admits eight Noether symmetries and corresponding eight conserved quantities:

$$
\begin{aligned}
\Gamma_1-\Gamma_5 \quad &\Rightarrow \quad I_1 = kq_1\dot{q}_2 - t\dot{q}_2 + q_2 - kq_2\dot{q}_1 \\
\Gamma_6 \quad &\Rightarrow \quad I_6 = m\dot{q}_2 \cos\left(\sqrt{\frac{k}{m}}t\right) + q_2\sqrt{\frac{m}{k}} \sin\left(\sqrt{\frac{k}{m}}t\right) \\
\Gamma_7 \quad &\Rightarrow \quad I_7 = -q_2\sqrt{\frac{m}{k}} \cos\left(\sqrt{\frac{k}{m}}t\right) + m\dot{q}_2 \sin\left(\sqrt{\frac{k}{m}}t\right) \\
\Gamma_8 \quad &\Rightarrow \quad I_8 = -\sqrt{\frac{m}{k}}\left(k^3\left(q_1^2+q_2^2\right) - m + k\left(2m\dot{q}_1 + t^2\right)\right. \\
&\quad \left. -k^2\left(m\left(\dot{q}_1^2+\dot{q}_2^2\right) + 2tq_1\right)\right)\left(1 - 2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right) \\
&\quad -2km\left(k(q_1+\dot{q}_1 t) - t - k^2(q_1\dot{q}_1 + q_2\dot{q}_2)\right)\sin\left(2\sqrt{\frac{k}{m}}t\right) \\
\Gamma_9 \quad &\Rightarrow \quad I_9 = \sqrt{\frac{m}{k}}\left(k^3\left(q_1^2+q_2^2\right) - m + k\left(2m\dot{q}_1 + t^2\right)\right. \\
&\quad \left. -k^2\left(m\left(\dot{q}_1^2+\dot{q}_2^2\right) + 2tq_1\right)\right)\sin\left(\sqrt{\frac{k}{m}}t\right)\cos\left(\sqrt{\frac{k}{m}}t\right) \\
&\quad +km\left(k(q_1+\dot{q}_1 t) - t - k^2(q_1\dot{q}_1 + q_2\dot{q}_2)\right)\left(1 - 2\sin^2\left(\sqrt{\frac{k}{m}}t\right)\right)
\end{aligned}
$$

$$\Gamma_{10} \quad \Rightarrow \quad I_{10} = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2} k(q_1^2 + q_2^2) - tq_1 - \frac{m}{k}\dot{q}_1 + \frac{t^2}{2k}$$

$$\Gamma_{14} \quad \Rightarrow \quad I_{14} = \sqrt{\frac{m}{k}}(kq_1 - t)\sin\left(\sqrt{\frac{k}{m}}t\right) + (k\dot{q}_1 - 1)\cos\left(\sqrt{\frac{k}{m}}t\right) m$$

$$\Gamma_{15} \quad \Rightarrow \quad I_{15} = \sqrt{\frac{m}{k}}(kq_1 - t)\cos\left(\sqrt{\frac{k}{m}}t\right) - (k\dot{q}_1 - 1)\sin\left(\sqrt{\frac{k}{m}}t\right) m.$$

None admits $f =$ constant, e.g.:

$$\Gamma_6 \quad \Rightarrow \quad f = -q_2\sqrt{mk}\sin\left(\sqrt{\frac{k}{m}}t\right),$$

$$\begin{aligned} \Gamma_8 \quad \Rightarrow \quad f &= 2tq_1 - \frac{t^2}{2k} + \frac{m}{k^2} - k(q_1^2 + q_2^2) \\ &-2\sqrt{\frac{m}{k}}(kq_1 - t)\cos\left(\sqrt{\frac{k}{m}}t\right)\sin\left(\sqrt{\frac{k}{m}}t\right) \\ &-\left(4tq_1 - \frac{t^2}{k} + \frac{m}{k^2} - 2k(q_1^2 + q_2^2)\right)\sin^2\left(\sqrt{\frac{k}{m}}t\right). \end{aligned} \tag{6}$$

Further details on system (5) in *MCN, Phys. Lett. A, 2011*.

Lagrange vindicated

In the Avertissement to his "Méchanique Analitique" (1788) Joseph-Louis Lagrange (1736-1813) wrote:

The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field. (tr. by J.R. Maddox:)

Lagrange vindicated

In the Avertissement to his "Méchanique Analitique" (1788) Joseph-Louis Lagrange (1736-1813) wrote:

The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field. (tr. by J.R. Maddox:)

It is a joke, isn't it??!!