FANTASTIC SYMMETRIES AND WHERE TO FIND THEM

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Lecture 4: Noether symmetries II.

- The Ostrogradsky's method for constructing Lagrangians for equations of order greater than two.
- Ghost-free quantization via symmetry preservation: Pais–Uhlenbeck model and its "ghosts".
- Ghost-free quantization via symmetry preservation: Higgs model with a complex ghost pair.

Generating ghosts

Some simple linear equations of classical mechanics yield serious problems when quantization à la Dirac is undertaken since states with negative norm, commonly called ghosts, appear. We present a "ghostbuster" based on the preservation of Lie symmetries of the original classical equations.

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fourth-order field-theoretic model of Pais-Uhlenbeck:

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Karl Jansenn, Julius Kuti and Chuan Liu, The Higgs model with a complex ghost pair, *Phys. Lett. B 309* 119-126 (1993)

$$\frac{1}{M^4}x^{(vi)} + \frac{2}{M^2}\left(\cos(2\Theta) + \frac{\omega^2}{2M^2}\right)x^{(iv)} - \left(1 + \frac{2\omega^2}{M^2}\cos(2\Theta)\right)x'' + \omega^2 x = 0$$

Higher order Lagrangians

$$L\left(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n)}\right)$$

$$0 = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \ldots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{\partial L}{\partial x^{(n)}}\right)$$

$$H = ??$$

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1801 – 1861

Mikhail Vasilevich Ostrogradsky, Mémoire sur le calcul des variations des intégrales multiples, *Journal für die reine und angewandte Mathematik 15* (1836) 332-354

Enter Ostrogradsky

Ostrogradsky defines the momenta as

$$p_1 = \frac{\partial L}{\partial \dot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{x}}\right) + \ldots + (-1)^{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left(\frac{\partial L}{\partial x^{(n)}}\right)$$

$$p_2 = \frac{\partial L}{\partial \ddot{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dddot{x}}\right) + \ldots + (-1)^{n-2}\frac{\mathrm{d}^{n-2}}{\mathrm{d}t^{n-2}}\left(\frac{\partial L}{\partial x^{(n)}}\right)$$

$$\vdots$$

$$p_n = \frac{\partial L}{\partial x^{(n)}}$$

the canonical coordinates according to

$$q_1 = x,\ q_2 = \dot{x},\ \ldots,\ q_n = x^{(n-1)}$$

and finally the Hamiltonian function is

$$H = -L + p_1 q_2 + p_2 q_3 + \ldots + p_{n-1} q_n + p_n x^{(n)}$$

Pais-Uhlenbeck model

Using the prescription of Ostrogradsky:

$$H = -\tfrac{1}{2}\gamma \left\{ \frac{p_2^2}{\gamma^2} - \left(\Omega_1^2 + \Omega_2^2\right) q_2^2 + \Omega_1^2 \Omega_2^2 q_1^2 \right\} + p_1 q_2$$

$$\dot{q}_1 = q_2 \qquad \dot{p}_1 = \gamma \Omega_1 \Omega_2 q_1$$

$$\dot{q}_2 = -\frac{p_2}{\gamma} \qquad \dot{p}_2 = -\gamma \left(\Omega_1^2 + \Omega_2^2\right) q_2 - p_1.$$

Carl M. Bender and Philip D. Mannheim, Giving up the ghost, *J. Phys. A: Math. Theor. 41* 304018 (2008)

"Ghost states are quantum states having negative norm. If a quantum theory has ghost states, it is fundamentally unacceptable because the norm of a quantum state is interpreted as a probability, and a negative probability is forbidden on physical grounds."

Is $\mathcal{PT}$ quantum theory necessary?

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Maybe $\mathcal{PT}$ is not necessary MCN & Leach, JMathPhys 2009.

Begin with the Lagrangian

$$L = \frac{1}{2}\left\{\ddot{z}^2 - \left(\Omega_1^2 + \Omega_2^2\right)\dot{z}^2 + \Omega_1^2\Omega_2^2 z^2\right\} + \frac{\mathrm{d}}{\mathrm{d}t}F(t, z, \dot{z})$$

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Six-dimensional Lie point symmetry algebra:

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = z\partial_z, \qquad \Gamma_3 = \cos(\Omega_1 t)\partial_z, \qquad \Gamma_4 = -\sin(\Omega_1 t)\partial_z,$$
$$\Gamma_5 = \cos(\Omega_2 t)\partial_z, \qquad \Gamma_6 = -\sin(\Omega_2 t)\partial_z,$$

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$$\Gamma_5 = \cos(\Omega_2 t)\partial_z, \quad \Gamma_6 = -\sin(\Omega_2 t)\partial_z,$$

and five Noether point symmetries with five first integrals

$$\Gamma_1 \Longrightarrow I_1 = \tfrac{1}{2}(-\Omega_1^2\Omega_2^2 z^2 - \Omega_1^2\dot{z}^2 - \Omega_2^2\dot{z}^2 - 2\dot{z}\dddot{z} + \ddot{z}^2)$$

$$\Gamma_3 \Longrightarrow I_3 = (\Omega_2^2 z + \ddot{z})\sin(\Omega_1 t)\Omega_1 + (\Omega_2^2\dot{z} + \dddot{z})\cos(\Omega_1 t)$$

$$\Gamma_4 \Longrightarrow I_4 = (\Omega_2^2 z + \ddot{z})\cos(\Omega_1 t)\Omega_1 - (\Omega_2^2\dot{z} + \dddot{z})\sin(\Omega_1 t)$$

$$\Gamma_5 \Longrightarrow I_5 = (\Omega_1^2 z + \ddot{z})\sin(\Omega_2 t)\Omega_2 + (\Omega_1^2\dot{z} + \dddot{z})\cos(\Omega_2 t)$$

$$\Gamma_6 \Longrightarrow I_6 = (\Omega_1^2 z + \ddot{z})\cos(\Omega_2 t)\Omega_2 - (\Omega_1^2\dot{z} + \dddot{z})\sin(\Omega_2 t).$$

Begin with the Lagrangian

$$L = \tfrac{1}{2}\left\{\ddot{z}^2 - \left(\Omega_1^2 + \Omega_2^2\right)\dot{z}^2 + \Omega_1^2\Omega_2^2 z^2\right\} + \frac{\mathrm{d}}{\mathrm{d}t}F(t, z, \dot{z})$$

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and five Noether point symmetries with five first integrals

$$\Gamma_1 \quad\Longrightarrow\quad I_1 = \tfrac{1}{2}(-\Omega_1^2\Omega_2^2 z^2 - \Omega_1^2\dot{z}^2 - \Omega_2^2\dot{z}^2 - 2\dot{z}\,\dddot{z} + \ddot{z}^2)$$

$$\Gamma_3 \quad\Longrightarrow\quad I_3 = (\Omega_2^2 z + \ddot{z})\sin(\Omega_1 t)\Omega_1 + (\Omega_2^2\dot{z} + \dddot{z})\cos(\Omega_1 t)$$

$$\Gamma_4 \quad\Longrightarrow\quad I_4 = (\Omega_2^2 z + \ddot{z})\cos(\Omega_1 t)\Omega_1 - (\Omega_2^2\dot{z} + \dddot{z})\sin(\Omega_1 t)$$

$$\Gamma_5 \quad\Longrightarrow\quad I_5 = (\Omega_1^2 z + \ddot{z})\sin(\Omega_2 t)\Omega_2 + (\Omega_1^2\dot{z} + \dddot{z})\cos(\Omega_2 t)$$

$$\Gamma_6 \quad\Longrightarrow\quad I_6 = (\Omega_1^2 z + \ddot{z})\cos(\Omega_2 t)\Omega_2 - (\Omega_1^2\dot{z} + \dddot{z})\sin(\Omega_2 t).$$

$$I_{\mathrm{aut}} = \tfrac{1}{2}(I_3^2 + I_4^2 + I_5^2 + I_6^2)$$

Right Hamiltonian

That implies

$$I_{\mathrm{aut}} = \frac{1}{2}\left[(\Omega_2^2 z + \ddot{z})^2\Omega_1^2 + (\Omega_2^2\dot{z} + \dddot{z})^2 + (\Omega_1^2 z + \ddot{z})^2\Omega_2^2 + (\Omega_1^2\dot{z} + \dddot{z})^2\right]$$

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We make the obvious transformations

$$q_1 = \ddot{z} + \Omega_2^2 z, \qquad q_2 = \ddot{z} + \Omega_1^2 z,$$
$$p_1 = \dddot{z} + \Omega_2^2 \dot{z}, \qquad p_2 = \dddot{z} + \Omega_1^2 \dot{z},$$

and consequently we obtain the Hamiltonian

Right Hamiltonian

That implies

$$I_{\rm aut} = \frac{1}{2}\left[(\Omega_2^2 z + \ddot{z})^2\Omega_1^2 + (\Omega_2^2\dot{z} + \dddot{z})^2 + (\Omega_1^2 z + \ddot{z})^2\Omega_2^2 + (\Omega_1^2\dot{z} + \dddot{z})^2\right]$$

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$$H \equiv I_{\rm aut} = \frac{1}{2}[p_1^2 + p_2^2 + \Omega_1^2 q_1^2 + \Omega_2^2 q_2^2]$$

Right Hamiltonian

That implies

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We make the obvious transformations

$$q_1 = \ddot{z} + \Omega_2^2 z, \qquad q_2 = \ddot{z} + \Omega_1^2 z,$$
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and consequently we obtain the Hamiltonian

$$H \equiv I_{\rm aut} = \frac{1}{2}[p_1^2 + p_2^2 + \Omega_1^2 q_1^2 + \Omega_2^2 q_2^2]$$

and the corresponding canonical equations

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = -\Omega_1^2 q_1, \quad \dot{p}_2 = -\Omega_2^2 q_2.$$

This is the right Hamiltonian for the quantization of the fourth-order field-theoretic model of Pais-Uhlenbeck.

Preservation of symmetries

The application of the Legendre transformation gives

$$L = \frac{1}{2}[\dot{q}_1^2 + \dot{q}_2^2 - (\Omega_1^2 q_1^2 + \Omega_2^2 q_2^2)]$$

and corresponding Lagrange equations

$$\ddot{q}_1 = -\Omega_1^2 q_1, \qquad \ddot{q}_2 = -\Omega_2^2 q_2$$

which admit a seven-dimensional Lie point symmetry algebra

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = q_1\partial_{q_1}, \quad \Gamma_3 = \cos(\Omega_1 t)\partial_{q_1}, \quad \Gamma_4 = -\sin(\Omega_1 t)\partial_{q_1},$$
$$\Gamma_5 = \cos(\Omega_2 t)\partial_{q_2}, \quad \Gamma_6 = -\sin(\Omega_2 t)\partial_{q_2}, \quad \Gamma_7 = q_2\partial_{q_2}$$

and *five* Noether point symmetries with five first integrals

$$\begin{aligned}
\Gamma_1 &\implies I_1 = \frac{1}{2}(\Omega_1^2 q_1^2 + \Omega_2^2 q_2^2 + \dot{q}_1^2 + \dot{q}_2^2)\\
\Gamma_3 &\implies I_3 = \sin(\Omega_1 t)\Omega_1 q_1 + \cos(\Omega_1 t)\dot{q}_1\\
\Gamma_4 &\implies I_4 = \cos(\Omega_1 t)\Omega_1 q_1 - \sin(\Omega_1 t)\dot{q}_1\\
\Gamma_5 &\implies I_5 = \sin(\Omega_2 t)\Omega_2 q_2 + \cos(\Omega_2 t)\dot{q}_2\\
\Gamma_6 &\implies I_6 = \cos(\Omega_2 t)\Omega_2 q_2 - \sin(\Omega_2 t)\dot{q}_2.
\end{aligned}$$

Higgs model

$$x^{(vi)} = -2M^2\left(\cos(2\Theta) + \frac{\omega^2}{2M^2}\right)x^{(iv)} - M^4\left(1 + \frac{2\omega^2}{M^2}\cos(2\Theta)\right)x'' - M^4\omega^2 x$$

$$L = \frac{1}{2M^4}\Big(-2\cos(2\Theta)M^2\omega^2 x'^2 - 2\cos(2\Theta)M^2 x''^2 - M^4\omega^2 x^2 - M^4 x'^2 - \omega^2 x''^2 + x'''^2\Big)$$

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$$x^{(vi)} = -(\omega_1^2+\omega_2^2+\omega_3^2)x^{(iv)} - (\omega_1^2\omega_2^2+\omega_1^2\omega_3^2+\omega_2^2\omega_3^2)x'' - \omega_1^2\omega_2^2\omega_3^2 x$$

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$$L = \frac{1}{2}\Big(x'''^2 - (\omega_1^2+\omega_2^2+\omega_3^2)x''^2 + (\omega_1^2\omega_2^2+\omega_1^2\omega_3^2+\omega_2^2\omega_3^2)x'^2 - \omega_1^2\omega_2^2\omega_3^2 x^2\Big)$$

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$$x^{(vi)} = -(\omega_1^2+\omega_2^2+\omega_3^2)x^{(iv)} - (\omega_1^2\omega_2^2+\omega_1^2\omega_3^2+\omega_2^2\omega_3^2)x'' - \omega_1^2\omega_2^2\omega_3^2 x$$

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Eight-dimensional Lie point symmetry algebra:

$$\Gamma_1 = \sin(\omega_1 t)\partial_x, \Gamma_2 = \cos(\omega_1 t)\partial_x, \Gamma_3 = \sin(\omega_3 t)\partial_x, \Gamma_4 = \cos(\omega_3 t)\partial_x,$$
$$\Gamma_5 = \sin(\omega_2 t)\partial_x, \Gamma_6 = \cos(\omega_2 t)\partial_x, \Gamma_7 = x\partial_x, \Gamma_8 = \partial_t,$$

and seven Noether point symmetries with seven first integrals:

$$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6, \Gamma_8$$

Right Hamiltonian

$$I_{aut} = \frac{1}{2}(Int_1^2 + Int_2^2 + Int_3^2 + Int_4^2 + Int_5^2 + Int_6^2)$$

that implies the obvious transformations

$$\begin{aligned}
q_1 &= \omega_2^2\omega_3^2 x + (\omega_2^2 + \omega_3^2)x'' + x^{(iv)} \\
q_2 &= \omega_1^2\omega_3^2 x + (\omega_1^2 + \omega_3^2)x'' + x^{(iv)} \\
q_3 &= \omega_1^2\omega_2^2 x + (\omega_1^2 + \omega_2^2)x'' + x^{(iv)} \\
p_1 &= \omega_2^2\omega_3^2 x' + (\omega_2^2 + \omega_3^2)x''' + x^{(v)} \\
p_2 &= \omega_1^2\omega_3^2 x' + (\omega_1^2 + \omega_3^2)x''' + x^{(v)} \\
p_3 &= \omega_1^2\omega_2^2 x' + (\omega_1^2 + \omega_2^2)x''' + x^{(v)}
\end{aligned}$$

and consequently we obtain the Hamiltonian

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q_3 &= \omega_1^2\omega_2^2 x + (\omega_1^2 + \omega_2^2)x'' + x^{(iv)} \\
p_1 &= \omega_2^2\omega_3^2 x' + (\omega_2^2 + \omega_3^2)x''' + x^{(v)} \\
p_2 &= \omega_1^2\omega_3^2 x' + (\omega_1^2 + \omega_3^2)x''' + x^{(v)} \\
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p_3 &= \omega_1^2\omega_2^2 x' + (\omega_1^2 + \omega_2^2)x''' + x^{(v)}
\end{aligned}$$

and consequently we obtain the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2 + \omega_3^2 q_3^2)$$

This is the right Hamiltonian for the quantization of the sixth-order Higgs model.

Preservation of symmetries

The application of the Legendre transformation gives

$$L = \frac{1}{2}[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - (\omega_1^2 q_1^2 + \omega_2^2 q_2^2 + \omega_3^2 q_3^2)]$$

and corresponding Lagrange equations

$$\ddot{q}_1 = -\omega_1^2 q_1, \qquad \ddot{q}_2 = -\omega_2^2 q_2, \qquad \ddot{q}_3 = -\omega_3^2 q_3$$

which admit a ten-dimensional Lie point symmetry algebra and *seven* Noether point symmetries with seven first integrals.

$$\begin{aligned} \Lambda_1 = \partial_t, \quad \Lambda_2 &= q_1\partial_{q_1}, \quad \Lambda_3 = \cos(\omega_1 t)\partial_{q_1}, \quad \Lambda_4 = -\sin(\omega_1 t)\partial_{q_1}, \\ \Lambda_5 &= q_2\partial_{q_2}, \quad \Lambda_6 = \cos(\omega_2 t)\partial_{q_2}, \quad \Lambda_7 = -\sin(\omega_2 t)\partial_{q_2}, \\ \Lambda_8 &= q_3\partial_{q_3}, \quad \Lambda_9 = \cos(\omega_3 t)\partial_{q_3}, \quad \Lambda_{10} = -\sin(\omega_3 t)\partial_{q_3}. \end{aligned}$$

More details in *MCN, Theor.Math.Phys., 2011.*

Eliminating the ghosts

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Sophus Lie and Emmy Noether