The connection between the geometry of manifold and the Davey – Stewartson equation

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The study of the connection between the differential geometry and nonlinear integrable equations is one of the topics of interest to mathematicians and physicists. The root of these studies goes back to the 19th century, to the works of the great geometers Li, Darboux, Bianchi, Backlund and etc. A new section of geometry, called soliton geometry, associated with integrable equations in (1+1)-dimensional space-time, has been investigated well, while in the space-time of (2+1)-dimensions these studies are still being discussed.

OUTLINE

1. Integrable (2+1)-dimensional Davey – Stewartson equation
2. Sym – Tafel formula
3. First and second fundamental forms for the (2+1)-dimensional Davey – Stewartson equation

In my talk, first, I will give some information about DSE and its integrability. Secondly, we will consider Sym – Tafel formula and derive some formulas that allow to rewrite poisition vector into its matrix form. Then, we establish the first and second fundamental forms for the (2+1)-dimensional Davey – Stewartson equation.

The (2+1)-dimensional Davey – Stewartson equation

The DSE has the form

$$iq_t + q_{xx} + q_{yy} - uq = 0 \tag{1}$$
$$ir_t - r_{xx} - r_{yy} - vr = 0 \tag{2}$$
$$u_{xx} + v_{yy} + (rq)_{xx} = 0 \tag{3}$$

where q and r are complex-valued functions, $r = \delta\bar{q}$, $\delta = \pm 1$, u, v are potentials.
Standard Lax Representation (LR) for the DSE reads as

$$\Phi_y = \sigma_3\Phi_x + Q\Phi \tag{4}$$
$$\Phi_t = 2i\sigma_3\Phi_{xx} + 2iV_1\Phi_x + iV_0\Phi \tag{5}$$

where

$$Q = \begin{pmatrix} 0 & q \\ \delta\bar{q} & 0 \end{pmatrix}, \quad V_1 = 2iQ = 2i\begin{pmatrix} 0 & q \\ \delta\bar{q} & 0 \end{pmatrix}, \tag{6}$$

$$V_0 = i\begin{pmatrix} |q|^2 + 2u & q_x + q_y \\ -\bar{q}_x + \bar{q}_y & -|q|^2 - 2v \end{pmatrix}. \tag{7}$$

For convinience we rewrite the LR (4)-(5) as

$$\Phi_x = U\Phi, \tag{8}$$

$$\Phi_y = W\Phi, \tag{9}$$

$$\Phi_t = V\Phi, \tag{10}$$

where

$$U = -i\lambda\sigma_3 + Q, \tag{11}$$

$$W = \sigma_3 U + Q, \tag{12}$$

$$V = 2i\sigma_3(U_x + U^2) + 2iQU + iV_0. \tag{13}$$

The Sym – Tafel formula

We consider the Sym – Tafel formula

$$r = \Phi^{-1}\Phi_{\lambda}, \tag{14}$$

where $r = \begin{pmatrix} r_3 & r^- \\ r^+ & -r_3 \end{pmatrix}$, $r^{\pm} = r_1 \pm ir_2$ is matrix analogy of the position vector, $\vec{r} = (r_1, r_2, r_3)$, $\vec{r}^2 = r_1^2 + r_2^2 + r_3^2$, Φ is a solution of the linear system corresponding to the DSE, λ is a real parameter. My goal is to find I and II forms for the DSE, using the STE (14).

$$I = g_{ij}dx^i dx^j, \tag{15}$$

$$II = b_{ij}dx^i dx^j. \tag{16}$$

$$\vec{r}(x,t) = (r_1, r_2, r_3) \in so(3) \longrightarrow r(x,t) = \begin{pmatrix} r_3 & r^- \\ r^+ & -r_3 \end{pmatrix} \in su(2)$$

$$r_x^2 = \begin{pmatrix} r_{3x}^2 + r_x^- r_x^+ & 0 \\ 0 & r_{3x}^2 + r_x^- r_x^+ = \\ = \left(r_{3x}^2 + r_x^- r_x^+\right) I = \left(r_{1x}^2 + r_{2x}^2 + r_{3x}^2\right) I, & \end{pmatrix}$$

$$\vec{r}_x^2 = \frac{1}{2}(r_x^2). \tag{17}$$

$$\vec{r}_x \cdot \vec{r}_t = \frac{1}{2}(r_x r_t) \tag{18}$$

$$\vec{r}_x^2 = \frac{1}{2}(r_x^2) \tag{19}$$

$$\vec{r}_t^2 = \frac{1}{2}(r_t^2) \tag{20}$$

$$\vec{r}_x \cdot \vec{r}_t = \frac{1}{2}(r_x r_t) \tag{21}$$

Now we are ready to express r via matrices U, V, W. Let us rewrite the I fundamental form as follows

$$I = \frac{1}{2}[tr\left(U_\lambda^2\right)dx^2 + 2tr\left(U_\lambda V_\lambda\right)dxdt + tr\left(V_\lambda^2\right)dt^2 + \tag{22}$$

$$+2tr(U_\lambda W_\lambda)dxdy + 2tr(W_\lambda V_\lambda)dydt + tr(W_\lambda^2)dy^2] \tag{23}$$

$$I = -2dx^2 - 8\lambda dxdt - (16 - 4|q|^2)dt^2 - 2dy^2. \tag{24}$$

$$II = \frac{1}{2}[tr(r_{xx} \cdot n)dx^2 + 2tr(r_{xt} \cdot n)dxdt + tr(r_{tt} \cdot n)dt^2 +$$

$$+2tr(r_{xy} \cdot n)dxdy + 2tr(r_{yt} \cdot n)dydt + tr(r_{yy} \cdot n)dy^2]. \tag{25}$$

$$n = \pm \frac{\Phi^{-1}[U_\lambda, V_\lambda]\Phi}{\sqrt{\frac{1}{2}tr([U_\lambda, V_\lambda]^2)}} \tag{26}$$

The Gauss-Weingarten equation is given as

$$\vec{r}_{xt} = \Gamma^1_{31}\vec{r}_x + \Gamma^3_{31}\vec{r}_t + b_{13}\vec{n}, \quad (27)$$

$$\vec{r}_{yt} = \Gamma^1_{32}\vec{r}_x + \Gamma^3_{32}\vec{r}_t, \quad (28)$$

$$\vec{r}_{tt} = \Gamma^1_{33}\vec{r}_x + \Gamma^2_{33}\vec{r}_y + \Gamma^3_{33}\vec{r}_t + b_{33}\vec{n}, \quad (29)$$

where

$$b_{13} = i\frac{1}{\sqrt{qr}}(2ir(q_y + (1+2i)q_x) - 2q(4\lambda r - ir_y + (2+i)r_x)), \quad (30)$$

$$b_{33} = \frac{1}{\sqrt{qr}}((4-8i)q^2r^2 + r(-q_t + 4\lambda(q_y + (1+2i)q_x)) + +q(4(v_1+v_2+2i\lambda^2)r + r_t + 4\lambda r_y - (4-8i)\lambda r_x)). \quad (31)$$

The metrics g_{ij}:

$$g_{ij} = \begin{pmatrix} -1 & 0 & -8\lambda \\ 0 & 1 & 0 \\ -8\lambda & 0 & -16\lambda^2 - 4qr \end{pmatrix}. \tag{32}$$

Cristoffel simbols take the following forms

$$\Gamma^1_{31} = \frac{4\lambda(qr)_x}{12\lambda^2 - qr}, \quad (33)$$

$$\Gamma^1_{32} = \frac{4\lambda(qr)_y}{12\lambda^2 - qr}, \quad (34)$$

$$\Gamma^1_{33} = \frac{4\lambda(qr)_t + 2(4\lambda^2 + qr)(qr)_x}{12\lambda^2 - qr}, \quad (35)$$

$$\Gamma^2_{33} = 2(qr)_y, \quad (36)$$

$$\Gamma^3_{31} = \frac{(qr)_x}{24\lambda^2 - 2qr}, \quad (37)$$

$$\Gamma^3_{32} = \frac{(qr)_y}{24\lambda^2 - 2qr}, \quad (38)$$

$$\Gamma^3_{33} = -\frac{(qr)_t + 8\lambda(qr)_x}{24\lambda^2 - 2qr}. \quad (39)$$

Conclusion

In this work, using the Sym-Tafel approach, we obtained I and II fundamental forms of the DSE. Also, the form of the GWE that describe the manifold linked to the DSE is found. Finding the connection between curvature and DSE is the purpose of our further investigation.

Thank you for attention!