Explicit Parameterizations of a Complementary Family of Non-Bending Surfaces

Vladimir Pulov[1] and Ivailo Mladenov[2]

[1]Department of Physics, Technical University of Varna

[2]Institute of Biophysics, Bulgarian Academy of Science

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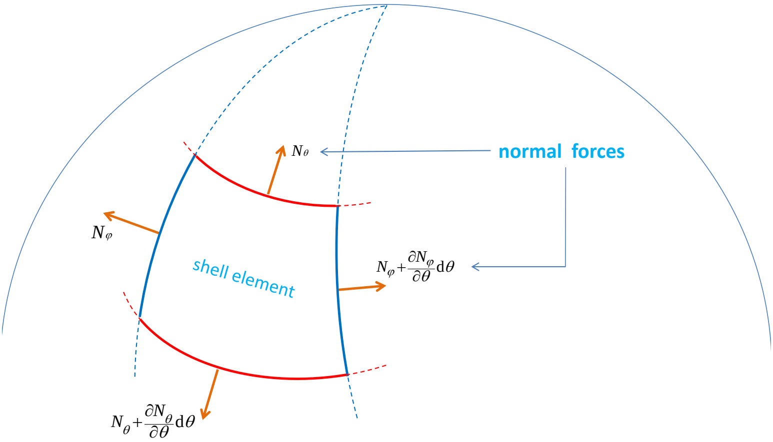
Outline

Shell Element is cut out by two meridians and two parallels
shearing and transverse forces are neglected



Non-Bending Condition
(Gurevich and Kalinin, 1981)

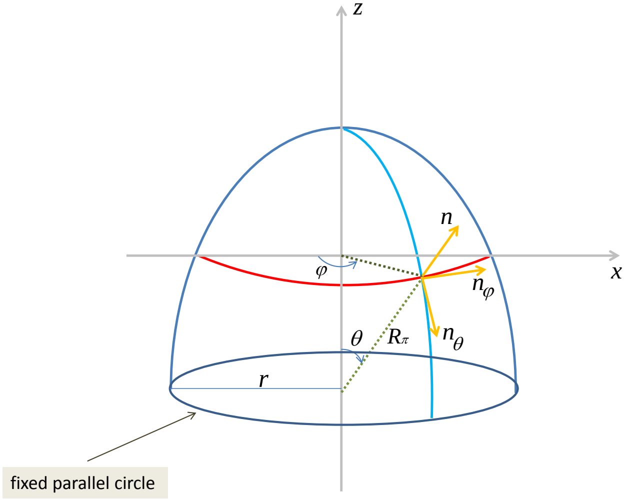
normals do not turn $=$ shell deforms without bending

$$\frac{\kappa_\mu}{\kappa_\pi} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}(N_\phi - \mathring{\nu} N_\theta) + (1 + \mathring{\nu})(N_\phi - N_\theta)\cot\theta = 0$$

κ_μ, κ_π – principal curvatures

N_ϕ, N_θ – normal (direct) forces

Middle Surface of a Shell of Revolution



Stress Equilibrium at a Fixed Parallel Circle
(Gurevich and Kalinin, 1981)

$$\mathring{N}_\theta = \frac{p}{2\mathring{\kappa}_\pi}, \qquad \mathring{N}_\phi = \frac{p}{2\mathring{\kappa}_\pi}\left(2 - \frac{\mathring{\kappa}_\mu}{\mathring{\kappa}_\pi}\right)$$

p – load

two parametric family of surfaces

represented via elliptic integral

Non-Bending Condition

in terms of principal curvatures κ_π, κ_μ

(Pulov and Mladenov, 2019)

$$\kappa_\mu = 2a\,\kappa_\pi^2 + 3\kappa_\pi, \qquad a = \frac{(\nu - 3)r}{2}$$

four classes of non-bending surfaces

in explicit canonical representations

r, ν – two parameters, $\nu = \frac{\mathring{\kappa}_\mu}{\mathring{\kappa}_\pi}$

Stress Disturbances at a Fixed Parallel Circle
(Gurevich, 1983)

$$\mathring{N}_\theta = \frac{p}{2\mathring{\kappa}_\pi}(1+\varepsilon), \qquad \mathring{N}_\phi = \frac{p}{2\mathring{\kappa}_\pi}\left(2 - \frac{\mathring{\kappa}_\mu}{\mathring{\kappa}_\pi}\right)$$

ε – stress disturbance (perturbation)

Non-Bending Condition $(\varepsilon \neq 0)$

$$\kappa_\mu = 2a\,\kappa_\pi^2 + \left(\frac{4a}{r} + \frac{\nu+3}{\nu-1}\right)\kappa_\pi + \frac{2a}{r^2}, \qquad a = \frac{(\nu+1)(\nu-3)r}{8(\nu-1)}$$

two parametric family of non-bending surfaces

$r,\ \nu$ – two real parameters, $\nu = \dfrac{\mathring{\kappa}_\mu}{\mathring{\kappa}_\pi}$

$\varepsilon = \dfrac{\nu-3}{\nu-1}r$ – stress disturbance

Complimentary Family of Non-Bending Surfaces
in the presence of stress disturbances at a fixed parallel circle

$$\kappa_\mu = 2a\,\kappa_\pi^2 + \left(\frac{4a}{r} + \frac{\nu+3}{\nu-1}\right)\kappa_\pi + \frac{2a}{r^2}, \qquad a = \frac{(\nu+1)(\nu-3)r}{8(\nu-1)}$$

$\nu = -1,$	$\kappa_\mu = -\kappa_\pi$	catenoid
$\nu = 0,$	$\kappa_\mu = 0,\ \kappa_\pi = \frac{1}{r}$	right circular cylinder
$\nu = 1,$	$\kappa_\mu = \kappa_\pi = \frac{1}{r}$	sphere
$\nu = 3,$	$\kappa_\mu = 3\kappa_\pi$	linear Weingarten surface $LW(2)$

Upper Right Branch

$$z(x) = \pm \frac{r}{2} \int_{(x/r)^2}^{1} \frac{((\nu+1)t - \nu + 3)\mathrm{d}t}{\sqrt{(1-t)\left((\nu+1)^2 t^2 - 2(\nu^2 - 6\nu + 1)t + (\nu+1)^2\right)}}$$

plus sign – surfaces inside the cylinder $\nu = 0$ (for $\nu > 0$)

minus sign – surfaces outside the cylinder $\nu = 0$ (for $\nu < 0$, $\nu \neq -1$)

r – radius of a fixed parallel circle

ν – free parameter

Polynomial Under the Radical

$$P(t) = (\nu + 1)^2 t^2 - 2(\nu^2 - 6\nu + 1)t + (\nu + 1)^2$$

Roots

$$\sigma = \frac{1 - 6\nu + \nu^2 + 4(1-\nu)\sqrt{-\nu}}{(1+\nu)^2}, \quad \tau = \frac{1 - 6\nu + \nu^2 - 4(1-\nu)\sqrt{-\nu}}{(1+\nu)^2}$$

Using Only Elementary Functions

$\nu = 0$ multiple roots $\sigma = \tau = 1$ (right circular cylinder)

$\nu = 1$ multiple roots $\sigma = \tau = -1$ (sphere)

Open Surfaces Outside the Cylinder $\nu = 0$
(profile curve, $\nu < 0, \quad \nu \neq -1$)

$$\eta(u) = \sqrt{\sigma_1 - 1}\,\mathrm{cn}(u, k_1), \qquad k_1 = \sqrt{\frac{\sigma_1 - 1}{\sigma_1 - \tau_1}}, \qquad u \in [-K(k_1), 0]$$

$$z_1(u) = \frac{r}{\sqrt{\sigma_1 - \tau_1}}\left(\lambda_1 F(\mathrm{am}(u, k_1), k_1) + \mu_1 E(\mathrm{am}(u, k_1), k_1)\right)$$

$$\lambda_1 = \tau + \frac{3-\nu}{1+\nu}, \qquad \mu_1 = \sigma_1 - \tau_1$$

$$x(u) = r\sqrt{1 + \eta(u)^2}, \qquad z(u) = z_1(u) - z_1(-K(k_1))$$

Open Surfaces Outside the Cylinder $\nu = 0$
(profile curve, $\nu < 0, \quad \nu \neq -1$)

$$z_1(x) = \frac{r}{\sqrt{\sigma_1 - \tau_1}} \Bigg[\lambda_1 F(\varphi(x), k_1) + \mu_1 E(\varphi(x), k_1) - \frac{1}{r}\sqrt{\frac{(\tau_1 - \sigma_1)(x^2 - r^2)(x^2 - \sigma_1 r^2)}{x^2 - \tau_1 r^2}} \Bigg]$$

$$\lambda_1 = \tau + \frac{3-\nu}{1+\nu}, \qquad \mu_1 = \sigma_1 - \tau_1, \qquad k_1 = \sqrt{\frac{\sigma_1 - 1}{\sigma_1 - \tau_1}}$$

$$\varphi(x) = \arcsin\sqrt{\frac{(\sigma_1 - \tau_1)(x^2 - r^2)}{(\sigma_1 - 1)(x^2 - \tau_1 r^2)}}, \qquad x \in [r, \ r\sqrt{\sigma_1}]$$

$F(\varphi(x), k_1)$, $E(\varphi(x), k_1)$ – incomplete elliptic integrals

Open Profile Curve ($r = 1, \ \nu = -0.5$)

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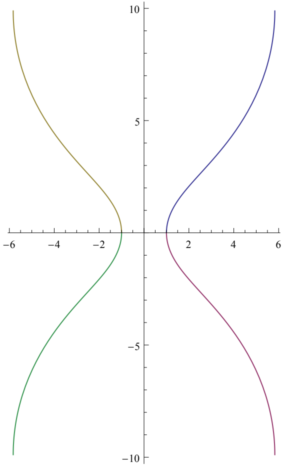

3D View $(r = 1, \ \nu = -0.5)$
Open Non-Bending Surface

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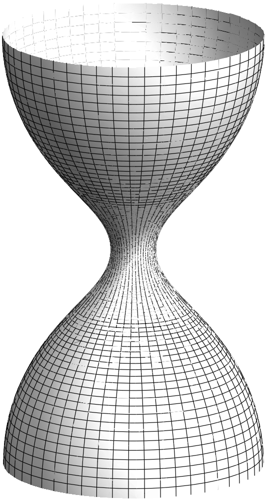

Open Part of 3D View ($r = 1$, $\nu = -0.5$)

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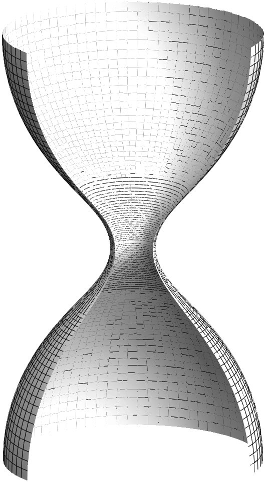
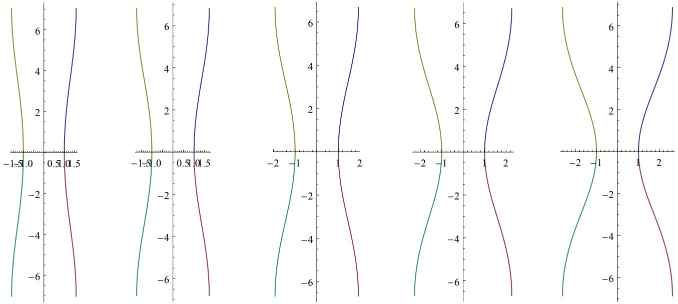


Open Profile Curves

$\nu = -\,0.05$ $\quad -0.07$ $\quad -0.1$ $\quad -0.15$ $\quad -0.2$

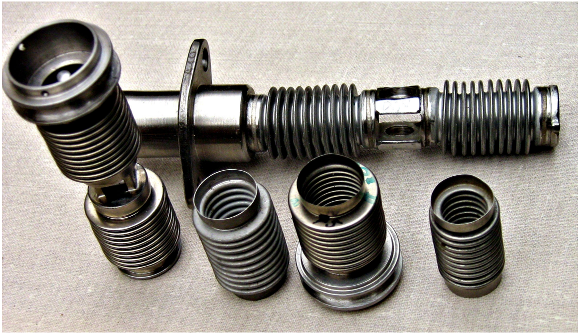


Metal Bellows (Sylphons)

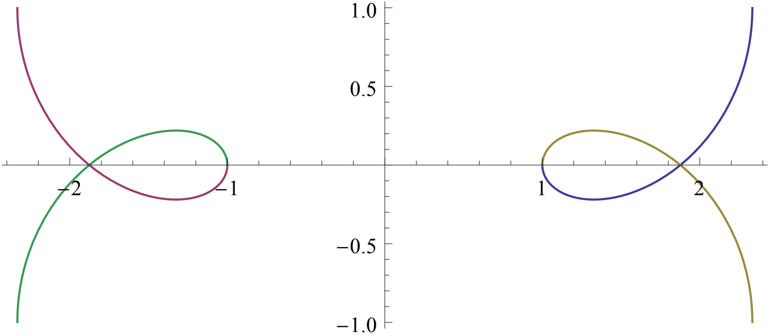
elastic vessels that can be compressed or extended under vacuum

- in pressure gauges of aggressive fluids to prevent leakage
- in pumps and valves as mechanical seals
- in exhaust gas pipes
- compensators of self vibrations and temperature differences

Metal Bellows (Sylphons)



Open Profile Curve $(r = 1,\ \nu = -6.25)$



Open Part of 3D View ($r = 1$, $\nu = -6.25$)

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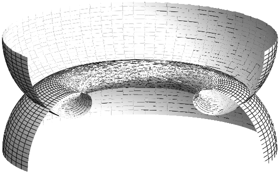
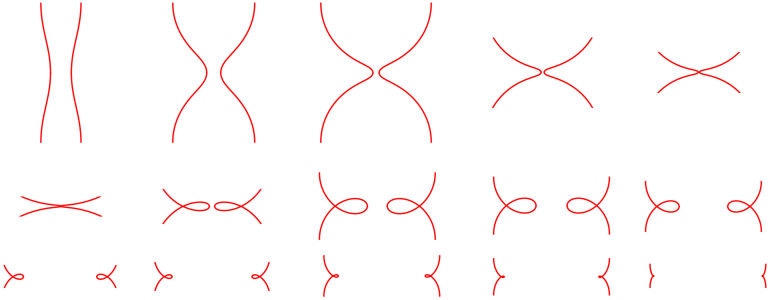


Open Profile Curves ($\nu < 0, \quad \nu \neq -1$)

$\nu = -0.1$	-0.5	-0.8	-0.9	-0.95
-0.99	-1.2	-2	-3	-10
-20	-30	-50	-100	-1000



Closed Surfaces Inside the Cylinder $\nu = 0$
(profile curve, $\nu > 0$)

$$z_2(x) = \frac{r}{2\sqrt{A}}\left[\left(\frac{4}{1+\nu} - A\right)F(\varphi(x), k_2) + 2A\left(E(\varphi(x), k_2) - \frac{\sin\varphi(x)\Delta(\varphi(x))}{1+\cos\varphi(x)}\right)\right]$$

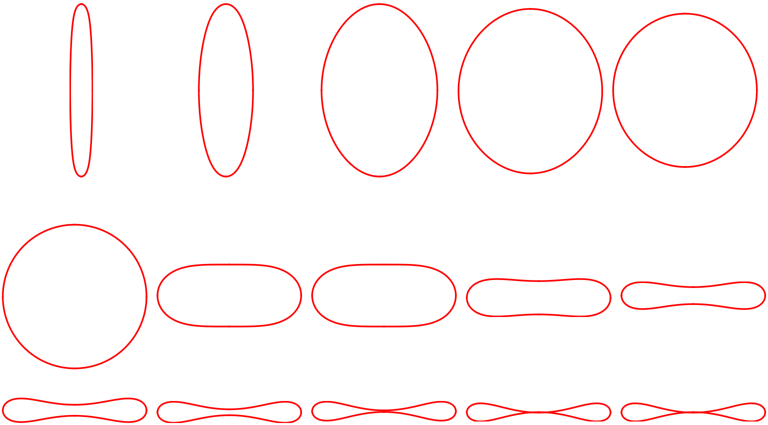
$$A = \frac{1}{2}\sqrt{(\sigma_2+\tau_2-2)^2 - (\sigma_2-\tau_2)^2}, \qquad k_2 = \frac{1}{2}\sqrt{2 - \frac{\sigma_2+\tau_2-2}{A}}$$

$$\varphi(x) = \arccos\left(\frac{A-1+(x/r)^2}{A+1-(x/r)^2}\right), \quad \Delta(\varphi(x)) = \sqrt{1-k_2^2\sin^2\varphi(x)}, \quad x \in [0, r]$$

$F(\varphi, k_2)$, $E(\varphi, k_2)$ – incomplete elliptic integrals

Closed Profile Curves ($\nu > 0$)

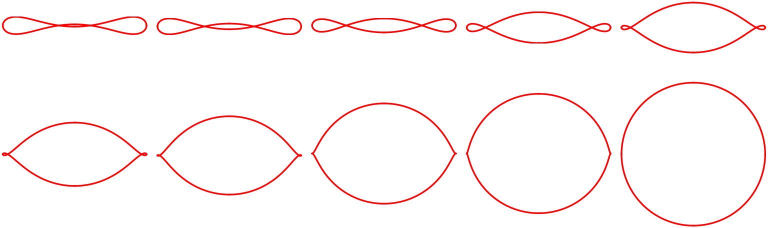
$\nu = 0.01$	0.1	0.25	0.5	0.8
1	2	3	5	7
8	9	10	10.5	11



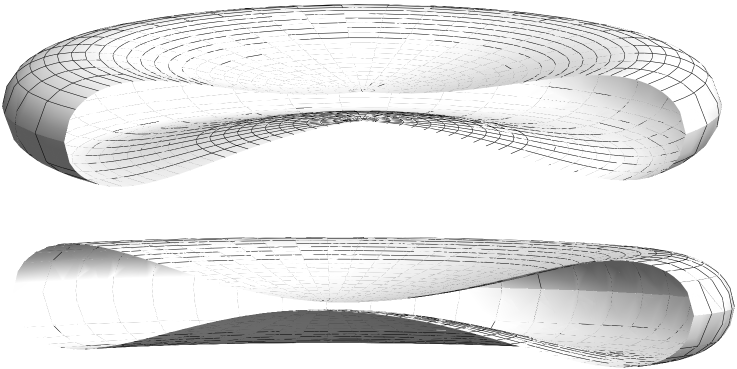
Closed Profile Curves ($\nu > 0$)

$\nu = 11$ 12 15 25 50

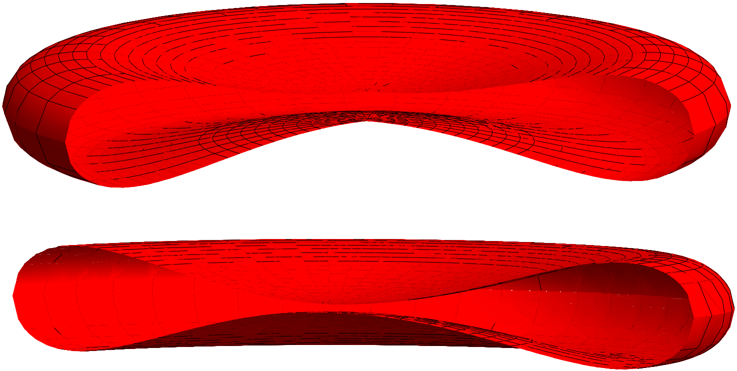
90 200 10^3 10^4 10^{10}



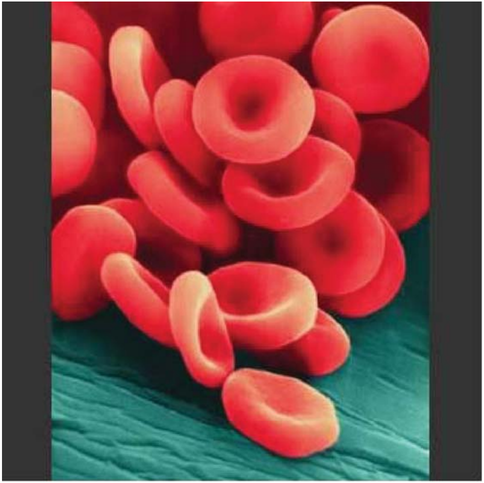
Open Part of 3D View ($r = 1$, $\nu = 9$ and $\nu = 10$)



Open Part of 3D View ($r = 1$, $\nu = 9$ and $\nu = 10$)



Red Blood Cells



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