Nonlocal complex modified Korteweg-de Vries equation and its exact solutions

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Introduction and motivation

In [Ablowitz M J and Musslimani Z H 2013 Integrable nonlocal nonlinear Schrodinger equation, Phys. Rev. Lett. 110, 064105], Ablowitz and Musslimani introduced the nonlocal nonlinear Schrodinger equation and got its explicit solutions by inverse scattering. Quite a lot of work were done after that for this equation and other equations:

- M.J. Ablowitz and Z.H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrodinger equation, Nonlinearity 29, 915–946, (2016).
- M. Gurses and A. Pekcan, Sym, Diff Eq and Applic 266, 27, (2018).
- Fokas A S Integrable multidimensional versions of the nonlocal nonlinear Schrodinger equation, Nonlinearity 29, 319–324, (2016) and et.all

1. (1+1)-dimensional nonlocal complex modified Korteweg-de Vries equation

Nonlocal complex modified Korteweg-de Vries system of equation has the form

$$\begin{aligned} q_t(x,t) + q_{xxx}(x,t) - 6q(x,t)r(x,t)q_x(x,t) = 0, \\ r_t(x,t) + r_{xxx}(x,t) - 6q(x,t)r(x,t)r_x(x,t) = 0, \end{aligned}$$

where $q(x,t)$ is a complex function. Under the symmetry reduction $r(x,t) = \sigma q^*(-x-t), \sigma = \mp 1$ we can get nonlocal cmKdV equation [Ablowitz M J and Musslimani Z H, Nonlinearity 29 915 (2016), M. Gurses and A. Pekcan, Sym, Diff Eq and Applic 266, 27, (2018)].

2. Ablowiz-Musslimani type of reduction

(1+1)-dimensional

Reverse space-time: $r(x,t) = \sigma q^*(-x,-t)$, $\sigma = \mp 1$
Reverse time: $r(x,t) = \sigma q^*(x,-t)$, $\sigma = \mp 1$
Reverse-space: $r(x,t) = \sigma q^*(-x,t)$, $\sigma = \mp 1$

(2+1)-dimensional

Reverse space-time: $r(x,t) = \sigma q^*(-x,-y,-t)$, $\sigma = \mp 1$
Reverse time: $r(x,t) = \sigma q^*(x,y,-t)$, $\sigma = \mp 1$
Reverse-space: $r(x,t) = \sigma q^*(-x,-y,t)$, $\sigma = \mp 1$

3. (2+1)-dimensional nonlocal complex modified Korteweg-de Vries equation

Nonlocal complex modified Korteweg-de Vries system of equation has the form

$$iq_t + iq_{xxy} - vq + i(wq)_x = 0, \qquad (1)$$
$$v_x - 2i\delta(q^*_{xy}(-x,-y,-t)q - q^*(-x,-y,-t)q_{xy}) = 0, \qquad (2)$$
$$w_x - 2\delta(qq^*(-x,-y,-t))_y = 0, \qquad (3)$$

where $q(x,y,t)$ is a complex function, $v(x,y,t), w(x,y,t)$ are real functions. The symbol " $*$ " denotes the complex conjugate. Local (classical) system was presented in work: Symmetry 7(3) (2015) 1352-1375. by Ratbay Myrzakulov, Muthusamy Lakshmanan and et.all.

4. Lax pair

Nonlocal complex modified Korteweg-de Vries system of equations is integrable and admits a Lax pair. The corresponding Lax pair of equations (1)-(3) reads as

$$\begin{aligned} \Psi_x &= A\Psi, & (4) \\ \Psi_t &= 4\lambda^2\Psi_y + B\Psi, & (5) \end{aligned}$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad B = \lambda B_1 + B_0,$$

and $\Psi = (\psi_1(\lambda, x, y, t), \psi_2(\lambda, x, y, t))^T$ is the vector eigenfunction and T denotes the transpose of the matrix.

The matrices A_0, σ_3, B_0, B_1 are given by

$$
\begin{aligned}
B_0 &= -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & -q_{xy} - wq \\ r_{xy} + wr & 0 \end{pmatrix}, \\
B_1 &= iw\sigma_3 + 2i\sigma_3 A_{0y}, \\
A_0 &= \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{aligned}
$$

and $r = \delta q^*(-x, -y, -t)$, $\delta = \pm 1$.
The compatibility condition of equations (4)-(5) is

$$A_t - B_x + [A, B] - 4\lambda^2 A_y = 0,$$

where $[A, B] = AB - BA$. By direct calculation of above equation, we can yield the (2+1)-dimensional nonlocal cmKdV system of equations.

5. Nonlocal reductions

Case 1: $r(x,y,t) = \delta q^*(x,y,t)$
The (2+1)-dimensional local cmKdV system reads as

$$iq_t + iq_{xxy} - vq + i(wq)_x = 0, \tag{6}$$
$$v_x - 2i\delta(q^*_{xy}(x,y,t)q - q^*(x,y,t)q_{xy}) = 0, \tag{7}$$
$$w_x - 2\delta(qq^*(x,y,t))_y = 0. \tag{8}$$

Case 2: $r(x,y,t) = \delta q^*(x,y,-t)$
The (2+1)-dimensional nonlocal reverse time cmKdV system reads as

$$iq_t + iq_{xxy} - vq + i(wq)_x = 0, \tag{9}$$
$$v_x - 2i\delta(q^*_{xy}(x,y,-t)q - q^*(x,y,-t)q_{xy}) = 0, \tag{10}$$
$$w_x - 2\delta(qq^*(x,y,-t))_y = 0. \tag{11}$$

Case 3: $r(x,y,t) = \delta q^*(-x,-y,t)$
The (2+1)-dimensional nonlocal reverse space cmKdV system reads as

$$iq_t + iq_{xxy} - vq + i(wq)_x = 0, \quad (12)$$
$$v_x - 2i\delta(q^*_{xy}(-x,-y,t)q - q^*(-x,-y,t)q_{xy}) = 0, \quad (13)$$
$$w_x - 2\delta(qq^*(-x,-y,t))_y = 0. \quad (14)$$

Case 4: $r(x,y,t) = \delta q^*(-x,-y,-t)$
The (2+1)-dimensional nonlocal reverse space-time cmKdV system reads as

$$iq_t + iq_{xxy} - vq + i(wq)_x = 0, \quad (15)$$
$$v_x - 2i\delta(q^*_{xy}(-x,-y,-t)q - q^*(-x,-y,-t)q_{xy}) = 0, \quad (16)$$
$$w_x - 2\delta(qq^*(-x,-y,-t))_y = 0. \quad (17)$$

6. Darboux transformation

Firstly, one consider the transformation about linear function Ψ by

$$\Psi' = T\Psi = (\lambda I - S)\Psi, \tag{18}$$

where

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The DT (18) transforms the original Lax pair (4)-(5) into a new Lax pair. The new function Ψ' satisfies

$$\Psi'_x = A'\Psi', \tag{19}$$
$$\Psi'_t = 4\lambda^2\Psi'_y + B'\Psi', \tag{20}$$

where A' and B' depend on q', v', w' and λ.
In order to hold equation (19)-(20), the T must satisfies the system

$$T_x + TA = A'T, \tag{21}$$
$$T_t + TB = 4\lambda^2 T_y + B'T. \tag{22}$$

By substituting the expressions A', B', A, B and T in to the equation (21)-(22) and collecting the different powers of λ^i of the equation (21) we obtain the following set of identities

$$\lambda^0 \quad : \quad S_x = A_0' S - S A_0, \tag{23}$$

$$\lambda^1 \quad : \quad A_0' = A_0 + i[S, \sigma_3], \tag{24}$$

$$\lambda^2 \quad : \quad i I \sigma_3 = i \sigma_3 I. \tag{25}$$

The equation (22) gives us the following relations

$$\lambda^0 \quad : \quad -S_t = S B_0 - B_0' S, \tag{26}$$

$$\lambda^1 \quad : \quad B_0' = B_0 + B_1^{'} S - S B_1, \tag{27}$$

$$\lambda^2 \quad : \quad 4 S_y = B_1' - B_1. \tag{28}$$

Finally, after some calculation from systems (23)-(25) and (26)-(28) we can obtain the first iterated DT for the (2+1)-dimensional nonlocal cmKdV system as:

$$\begin{aligned} q' &= q - 2is_{12}, & (29)\\ r' &= r - 2is_{21}, & (30)\\ v' &= v + 4(s_{12}r_y - s_{21}q_y + 2is_{11}s_{11y} + 2is_{21}s_{12y}), & (31)\\ w' &= w - 4is_{11y} = w + 4is_{22y}, & (32) \end{aligned}$$

and additionally have $s_{21} = -s_{12}^*(-x,-y,-t)$, $s_{22} = s_{11}^*(-x,-y,-t)$.

We now assume that

$$S = H\Lambda H^{-1},$$

where

$$H = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \quad \Lambda \quad = \quad \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1^* \end{pmatrix},$$

where $(f_1, f_2)^T = (\psi_1(x, y, t), \psi_2(x, y, t))^T$ is a solution to equation (4)-(5) with $\lambda = \lambda_1$ and
$(g_1, g_2)^T = (-\psi_2^*(-x, -y, -t), \psi_1^*(-x, -y, -t))^T$ is the solution when $\lambda = -\lambda_1^*$.

Hence, the new solutions are written as

$$q' = q - 2is_{12}, \tag{33}$$

$$v' = v + 4(s_{12}r_y + s^*_{12}q_y + 2is_{11}s_{11y} - 2is^*_{12}s_{12y}), \tag{34}$$

$$w' = w - 4is_{11y}, \tag{35}$$

where

$$s_{11} = \frac{\lambda_1\psi_1\psi_1^*(-x,-y,-t) - \lambda_1^*\psi_2\psi_2^*(-x,-y,-t)}{\Delta},$$

$$s_{12} = \frac{(\lambda_1 + \lambda_1^*)\psi_1\psi_2^*(-x,-y,-t)}{\Delta}.$$

7. Exact solutions

Having the explicit form of the DT (33)-(35), we are ready to construct exact solutions of the nonlocal cmKdV (1)-(3). We assume trivial seed solutions as

$$q = v = w = 0.$$

Solving linear system (4)-(5) under the zero background, we can get the following solution:

$$\psi_1 = e^{-i\lambda_1 x + i\mu_1 y + 4i\lambda_1^2\mu_1 t + c}, \qquad (36)$$

$$\psi_2 = e^{i\lambda_1 x - i\mu_1 y - 4i\lambda_1^2\mu_1 t - c}. \qquad (37)$$

Then, the exact solutions of the (2+1)-dimensional nonlocal cmKdV system (1)-(3) are obtained by substituting the expression (36)-(37) in (33)-(35) with $\lambda = \alpha_1 + i\beta_1$, $\mu = \eta_1 + i\nu_1$, $c = c_R + ic_I$ as:

$$\begin{aligned} q' &= 2\beta_1 e^{2\theta_1} \sec[2\chi_1], \\ v' &= 16\beta_1(\alpha_1\nu_1 + \beta_1\eta_1) \sec^2[2\chi_1], \\ w' &= -8\beta_1\nu_1 \sec^2[2\chi_1], \end{aligned}$$

where

$$\begin{aligned} \theta_1 &= \beta_1 x - \nu_1 y - (8\eta_1\alpha_1\beta_1 + 4\nu_1(\alpha_1^2 - \beta_1^2))t + c_R, \\ \chi_1 &= -\alpha_1 x + \eta_1 y + (4\eta_1(\alpha_1^2 - \beta_1^2) - 8\nu_1\alpha_1\beta_1)t + c_I. \end{aligned}$$

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Conclusion and (some) open problems

- Lax pair of the nonlocal cmKdV is presented.
- Darboux transformation is constructed.
- Other type solutions?
- Properties for solutions?
- Conservation laws?

Thank you for your attention!