Curvature functionals, p-Willmore energy, and the p-Willmore flow

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Outline

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The Willmore energy

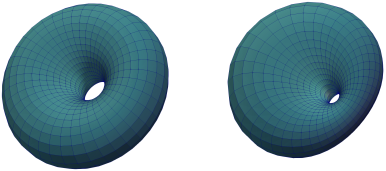
Let $\mathbf{R} : M \to \mathbb{R}^3$ be a smooth immersion of the closed surface M. Recall the Willmore energy functional

$$\mathcal{W}(M) = \int_M H^2 \, dS,$$

where H is the mean curvature of the surface.

Facts:

- Critical points of $\mathcal{W}(M)$ are called Willmore surfaces, and arise as natural generalizations of minimal surfaces.
- $\mathcal{W}(M)$ is invariant under reparametrizations, and less obviously under *conformal transformations of the ambient metric* (Mobius transformations of $\mathbb{R}^3$)



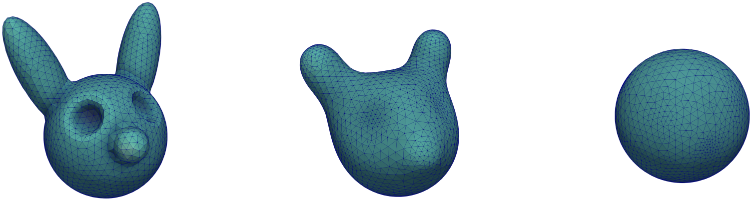
The Willmore energy (2)

From an aesthetic perspective, the Willmore energy produces surface fairing (i.e. smoothing). How to see this?

$$\frac{1}{4}\int_M (\kappa_1 - \kappa_2)^2 \, dS = \int_M (H^2 - K) \, dS = \mathcal{W}(M) - 2\pi\chi(M),$$

by the Gauss-Bonnet theorem.

Conclusion: The Willmore energy punishes surfaces for being non-umbilic!



Examples of Willmore-type energies

The Willmore energy arises frequently in mathematical biology, physics and computer vision – sometimes under different names.

- Helfrich-Canham energy,

$$E_H(M) := \int_M k_c(2H + c_0)^2 + \overline{k}K\, dS,$$

- Bulk free energy density,

$$\sigma_F(M) = \int_M 2k(2H^2 - K)\, dS,$$

- Surface torsion,

$$\mathcal{S}(M) = \int_M 4(H^2 - K)\, dS$$

When M is closed, all share critical surfaces with $\mathcal{W}(M)$.

General bending energy

More generally, these energies are all special cases of a model for bending energy proposed by Sophie Germain in 1820,

$$\mathcal{B}(M) = \int_M S(\kappa_1, \kappa_2)\, dS,$$

where S is a symmetric polynomial in κ_1, κ_2.

By Newton's theorem, this is equivalent to the functional

$$\mathcal{F}(M) = \int_M \mathcal{E}(H, K)\, dS,$$

where $\mathcal{E}$ is smooth in $H = \frac{1}{2}(\kappa_1 + \kappa_2), K = \kappa_1\kappa_2$.

Conclusion: Studying $\mathcal{F}(M)$ is natural from the point of view of bending energy, and reveals similarities between examples of scientific relevance.

General bending energy (2)

It is useful to study the functional $\mathcal{F}(M)$ on surfaces $M \subset \mathbb{M}^3(k_0)$ which are immersed in a 3-D *space form* of constant sectional curvature k_0.

Why leave Euclidean space?

- It's mathematically relevant (e.g. conformal geometry in S^3, geometry in the quaternions $\mathbb{H}$).
- Physicists care about immersions in "Minkowski space" which has constant sectional curvature -1. Can be modeled as $H^3 \cong \{q \in \mathbb{H}_H \,|\, qq^* = 1\}$ (hyperbolic quaternions).
- The notion of bending energy differs depending on the ambient space! For example, $(\kappa_1 - \kappa_2)^2 = 4(H^2 - K + k_0)$.

Particularly reasonable to study the *variations* of $\mathcal{F}(M)$, as they encode important geometric information about minimizing surfaces.

Framework for computing variations

Consider a variation of the surface M, i.e. a 1-parameter family of compactly supported immersions $\mathbf{r}(\mathbf{x}, t)$ as in the following diagram,

$$
\begin{array}{ccc}
 & & F_O\big(\mathbb{M}^3(k_0)\big) \\
 & \overset{\tilde{\mathbf{r}}}{\nearrow} & \big\downarrow \pi \\
M \times (-\varepsilon, \varepsilon) & \xrightarrow{\quad \mathbf{r} \quad} & \mathbb{M}^3(k_0)
\end{array}
$$

Choosing a local section $\{\mathbf{e}_J\}$ of $F_O\big(\mathbb{M}^3(k_0)\big)$ and a dual basis $\{\omega^I\}$ such that $\omega^I(\mathbf{e}_J) = \delta^I_J$, it follows that:

- Metric on $\mathbb{M}^3(k_0)$: $\quad g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2.$
- Connection on $\mathbb{M}^3(k_0)$: $\quad \nabla \mathbf{e}_I = \mathbf{e}_J \otimes \omega^J_I.$
- Volume form on $\mathbb{M}^3(k_0) = \omega^1 \wedge \omega^2 \wedge \omega^3.$

Connection is Levi-Civita (torsion-free) when $\omega^I_J = -\omega^J_I$.

Framework for computing variations (2)

The Cartan structure equations on $\mathbb{M}^3(k_0)$ are then

$$d\omega^I = -\omega^I_J \wedge \omega^J,$$
$$d\omega^I_J = -\omega^I_K \wedge \omega^K_J + \frac{1}{2} R^I_{JKL} \omega^K \wedge \omega^L.$$

We may assume the normal velocity of $\mathbf{r}$ satisfies

$$\frac{\partial \mathbf{r}}{\partial t} = u\,\mathbf{N},$$

for some smooth $u : M \times \mathbb{R} \to \mathbb{R}$. Pulling back the frame to $M \times \mathbb{R}$, we may further assume $\mathbf{e}_3 := \mathbf{N}$ is normal to $M \times \{t\}$ for each t, in which case

$$\overline{\omega}^i = \omega^i \qquad (i = 1, 2),$$
$$\overline{\omega}^3 = u\,dt.$$

General first variation

Using this, it is possible to compute the following necessary condition for criticality with respect to $\mathcal{F}$.

Theorem: Gruber, T., Tran

The first variation of the curvature functional $\mathcal{F}$ is given by

$$\delta \int_M \mathcal{E}(H,K)\, dS$$
$$= \int_M \left(\frac{1}{2}\mathcal{E}_H + 2H\mathcal{E}_K\right)\Delta u + \left((2H^2 - K + 2k_0)\mathcal{E}_H + 2HK\mathcal{E}_K - 2H\mathcal{E}\right)u$$
$$- \mathcal{E}_K \langle h, \mathrm{Hess}\, u\rangle \; dS,$$

where $\mathcal{E}_H, \mathcal{E}_K$ denote the partial derivatives of $\mathcal{E}$ with respect to H resp. K, and h is the shape operator of M ($\mathrm{II} = h\,\mathbf{N}$).

General second variation

Theorem: Gruber, T., Tran

At a critical immersion of M, the second variation of $\mathcal{F}$ is given by

$$\begin{aligned}
\delta^2 \int_M \mathcal{E}(H,K)\, dS &= \int_M \left(\frac{1}{4}\mathcal{E}_{HH} + 2H\mathcal{E}_{HK} + 4H^2\mathcal{E}_{KK} + \mathcal{E}_K\right)(\Delta u)^2\, dS \\
&+ \int_M \mathcal{E}_{KK}\langle h, \text{Hess}\, u\rangle^2\, dS - \int_M (\mathcal{E}_{HK} + 4H\mathcal{E}_{KK})\Delta u \langle h, \text{Hess}\, u\rangle\, dS \\
&+ \int_M \mathcal{E}_K\left(u\langle\nabla K, \nabla u\rangle - 3u\langle h^2, \text{Hess}\, u\rangle - 2\, h^2(\nabla u, \nabla u) - |\text{Hess}\, u|^2\right) dS \\
&+ \int_M \Big((2H^2 - K + 2k_0)\mathcal{E}_{HH} + 2H(4H^2 - K + 4k_0)\mathcal{E}_{HK} + 8H^2K\mathcal{E}_{KK} \\
&\qquad - 2H\mathcal{E}_H + (3k_0 - K)\mathcal{E}_K - \mathcal{E}\Big)u\Delta u\, dS \\
&+ \int_M \Big((2H^2 - K + 2k_0)^2\mathcal{E}_{HH} + 4HK(2H^2 - K + 2k_0)\mathcal{E}_{HK} + 4H^2K^2\mathcal{E}_{KK} \\
&\qquad - 2K(K - 2k_0)\mathcal{E}_K - 2HK\mathcal{E}_H + 2(K - 2k_0)\mathcal{E}\Big)u^2\, dS \\
&+ \int_M (2\mathcal{E}_H + 6H\mathcal{E}_K - 2(2H^2 - K + 2k_0)\mathcal{E}_{HK} - 4HK\mathcal{E}_{KK})u\langle h, \text{Hess}\, u\rangle\, dS \\
&+ \int_M (\mathcal{E}_H + 4H\mathcal{E}_K)h(\nabla u, \nabla u)\, dS + \int_M \mathcal{E}_H\, u\langle\nabla H, \nabla u\rangle\, dS \\
&- \int_M (2(K - k_0)\mathcal{E}_K + H\mathcal{E}_H)|\nabla u|^2\, dS,
\end{aligned}$$

where the subscripts $\mathcal{E}_{HH}, \mathcal{E}_{HK}, \mathcal{E}_{KK}$ denote the second partial derivatives of $\mathcal{E}$ in the appropriate variables.

Advantages of these variational results

- Valid in any space form of constant sectional curvature k_0.
- Quantities involved are as elementary as possible; directly computable from surface fundamental forms.
- Can be used to studying many specific functionals.

Example: these expressions immediately yield the known variation of the Willmore functional,

$$\delta \int_M H^2\, dS = \int_M \left(H\Delta u + 2H(H^2 - K + 2k_0)u \right) dS.$$

It follows that closed Willmore surfaces in $\mathbb{M}^3(k_0)$ are characterized by the equation

$$\Delta H + 2H(H^2 - K + 2k_0) = 0.$$

The p-Willmore energy

It is further interesting to consider the *p-Willmore energy*,

$$\mathcal{W}^p(M) = \int_M H^p \, dS, \qquad p \in \mathbb{Z}_{\geq 0}.$$

Notice that the Willmore energy is recovered as $\mathcal{W}^2$.

Why generalize Willmore?

- Conformal invariance is beautiful but very un-physical: unnatural for bending energy.
- $\mathcal{W}^0, \mathcal{W}^1$, and $\mathcal{W}^2$ are quite different. Are other $\mathcal{W}^p$ different?

We will see that the p-Willmore energy is highly connected to minimal surface theory when $p > 2$!!

Variations of p-Willmore energy

Corollary: Gruber, T., Tran

The first variation of $\mathcal{W}^p$ is given by

$$\delta \int_M H^p \, dS = \int_M \left[\frac{p}{2} H^{p-1} \Delta u + (2H^2 - K + 2k_0) p H^{p-1} u - 2H^{p+1} u \right] \, dS,$$

Moreover, the second variation of $\mathcal{W}^p$ at a critical immersion is given by

$$\begin{aligned}
\delta^2 \int_M H^p \, dS &= \int_M \frac{p(p-1)}{4} H^{p-2} (\Delta u)^2 \, dS \\
&+ \int_M p H^{p-1} \left(h(\nabla u, \nabla u) + 2u \langle h, \text{Hess } u \rangle + u \langle \nabla H, \nabla u \rangle - H |\nabla u|^2 \right) dS \\
&+ \int_M \left((2p^2 - 4p - 1) H^p - p(p-1) K H^{p-2} + 2p(p-1) k_0 H^{p-2} \right) u \Delta u \, dS \\
&+ \int_M \Big(4p(p-1) H^{p+2} - 2(p-1)(2p+1) K H^p + p(p-1) K^2 H^{p-2} \\
&\quad + 4(2p^2 - 2p - 1) k_0 H^p - 4p(p-1) k_0 K H^{p-2} + 4p(p-1) k_0^2 H^{p-2} \Big) u^2 \, dS.
\end{aligned}$$

Connection to minimal surfaces

In light of these variational results, define a *p-Willmore surface* to be any M satisfying the Euler-Lagrange equation,

$$\frac{p}{2}\Delta H^{p-1} - p(2H^2 - K + 2k_0)H^{p-1} + 2H^{p+1} = 0 \quad \text{on } M.$$

Using integral estimates inspired by Bergner and Jakob [1], it is possible to show the following:

Theorem: Gruber, T., Tran

When $p > 2$, any p-Willmore surface $M \subset \mathbb{R}^3$ satisfying $H = 0$ on ∂M is minimal.

More precisely, let $p > 2$ and $\mathbf{R} : M \to \mathbb{R}^3$ be an immersion of the p-Willmore surface M with boundary ∂M. If $H = 0$ on ∂M, then $H \equiv 0$ everywhere on M.

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Conclusion: $(p > 2)$-Willmore surface with $H = 0$ on $\partial M \iff$ minimal surface!

Sketch of Proof

- Use $\delta\mathcal{W}^p$, integration by parts, and geometric identities to establish the integral equality

$$\int_{\partial M} \nabla_{\mathbf{n}}(H^{p-1})\langle \mathbf{R}, \mathbf{N}\rangle = \int_{\partial M} H^{p-1}\left(\langle \nabla_{\mathbf{n}}\mathbf{N}, \mathbf{R}\rangle + (2/p)H\langle \nabla_{\mathbf{n}}\mathbf{R}, \mathbf{R}\rangle\right) + \frac{2(p-2)}{p}\int_M H^p,$$

  where $\mathbf{n}$ is conormal to the immersion $\mathbf{R}$ on ∂M.
- The condition $H \equiv 0$ on ∂M yields that

$$\int_M H^p \, dS = 0.$$

- The case of even p is obvious. When p is odd, separate M into regions where $H > 0$ and $H < 0$. Continuity implies that $H = 0$ on the boundaries, so the above equality applies. Conclude $H \equiv 0$ everywhere on M.

Consequences

This result has a number of interesting consequences. First,

- **Not** true for $p = 2$: many solutions (non-minimal catenoids, etc.) to Willmore equation with $H = 0$ on boundary.

Further, we see immediately:

Corollary: Gruber, T., Tran

There are no closed p-Willmore surfaces immersed in $\mathbb{R}^3$ when $p > 2$.

Proof. There are no closed minimal surfaces in $\mathbb{R}^3$.

In particular,

- The round sphere, Clifford torus, etc. are no longer minimizing in general for $\mathcal{W}^p$.
- A different minimization problem must be considered if there are to be closed solutions for all p.

Volume-constrained p-Willmore

Since $\mathcal{W}^p$ is physically motivated as a bending energy model, it is reasonable to consider its minimization subject to geometric constraints.

Let $M = \partial D$ and recall the volume functional

$$\mathcal{V} = \int_D dV = \int_{M\times[0,t]} \mathbf{R}^*(dV),$$

with first variation

$$\delta\mathcal{V} = \int_M u\, dS.$$

So, (by a Lagrange multiplier argument) M is a *volume-constrained p-Willmore surface* provided there is a constant C such that

$$\frac{p}{2}\Delta H^{p-1} - p(2H^2 - K + 2k_0)H^{p-1} + 2H^{p+1} = C.$$

Volume-constrained p-Willmore (2)

Why consider a volume constraint?

- Mimics the behavior of a lipid membrane in a solution with varying concentrations of solute.
- Acts as a "substitute" for conformal invariance; naturally limits the space of allowable surfaces.
- Allows for certain closed surfaces to be at least "almost stable".

Note the following result for spheres.

Theorem: Gruber, T., Tran

The round sphere $S^2(r)$ immersed in Euclidean space is **not** a stable local minimum of $\mathcal{W}^p$ under general volume-preserving deformations for each $p > 2$. More precisely, the bilinear index form is negative definite on the eigenspace of the Laplacian associated to the first eigenvalue, and it is positive definite on the orthogonal complement subspace.

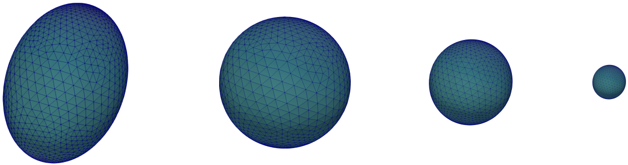
The p-Willmore flow

Let $V \subset \mathbb{R}^2$ and $X : V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a 1-parameter family of surface parametrizations, and let $\dot{X} = dX/dt$. To further investigate the p-Willmore energy, we now develop computational models for the *p-Willmore flow* of surfaces immersed in $\mathbb{R}^3$,

$$\dot{X} = -\delta \mathcal{W}^p(X).$$

We will consider two cases:

1. M is the graph of a smooth function $u : \mathbb{R}^2 \to \mathbb{R}$.
2. M is an abstract closed surface with identity map $u : M \to \mathbb{R}^3$.



Graphical model

First, we consider the case where M is given as the graph of a smooth function. Let:

- $M = \{(\mathbf{x}, u(\mathbf{x})) \,|\, \mathbf{x} \in \Omega\}$.
- I denote identity on $\mathbb{R}^3$.
- $A := \sqrt{\det g}$ denote the induced area element on M.
- $\mathbf{N} = (1/A)(\nabla u, -1)$ denote the "downward" unit normal on M.

It follows that the geometry on M can be expressed as,

$$g_{ij} = \delta_{ij} + u_i u_j, \qquad A = \sqrt{1 + |\nabla u|^2}, \qquad g^{ij} = \delta^{ij} - \frac{u^i u^j}{A^2},$$

$$\Delta_M = \frac{1}{A}\nabla \cdot \left(A\left(I - \frac{\nabla u \otimes \nabla u}{A^2}\right)\nabla\right), \qquad h_{ij} = \frac{u_{ij}}{A},$$

$$2H = \nabla \cdot \left(\frac{\nabla u}{A}\right), \qquad K = \frac{\det \nabla^2 u}{A^4}.$$

Graphical model (2)

The following is inspired by Deckelnick and Dziuk [2].

Problem: Graphical p-Willmore flow

Let $W := AH^{p-1}$. Given a surface M which is the graph of a smooth function u, find a family of surfaces $M(t) = \{(\mathbf{x}, u(\mathbf{x}, t)) \,|\, \mathbf{x} \in \Omega\}$ such that $M(0)$ is the graph of $u(\mathbf{x}, 0)$ and the p-Willmore flow equation

$$u_t + \frac{p}{2} A \nabla \cdot \left(\frac{1}{A} \left(I - \frac{\nabla u \otimes \nabla u}{A^2} \right) \nabla W \right) - A \nabla \cdot \left(WH \frac{\nabla u}{A^2} \right) = 0,$$

is satisfied for all $t \in [0, T]$. Alternatively, in weak form: find functions $u(\mathbf{x}, t)$ such that $M(t)$ is the graph of $u(\mathbf{x}, t)$, and the system of equations

$$\int_\Omega \frac{u_t}{A}\varphi - \left(\frac{p}{2A} \left(I - \frac{\nabla u \otimes \nabla u}{A^2} \right) \nabla W + \frac{WH}{A^2} \nabla u \right) \cdot \nabla \varphi = 0,$$

$$\int_\Omega 2H\psi + \left(\frac{\nabla u}{A} \right) \cdot \nabla \psi = 0,$$

$$\int_\Omega W\xi - AH^{p-1}\xi = 0.$$

is satisfied for all $t \in [0, T]$ and all $\varphi, \psi, \xi \in H^2$.

Properties of the p-Willmore flow: energy decrease

Theorem: Aulisa, Gruber

The graphical p-Willmore flow is energy-decreasing.

That is, given a family of surfaces $\{M(t)\}$ such that $M(t) = \{(\mathbf{x}, u(\mathbf{x}, t)) \,|\, \mathbf{x} \in \Omega\}$ and u_t obeys the p-Willmore flow equation

$$u_t + \frac{p}{2} A\nabla \cdot \left(\frac{1}{A} \left(I - \frac{\nabla u \otimes \nabla u}{A^2} \right) \nabla W \right) - A\nabla \cdot \left(WH \frac{\nabla u}{A^2} \right) = 0,$$

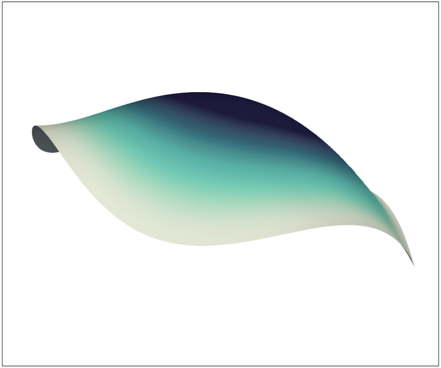
with $u = f$ and $H \equiv 0$ on ∂M, the p-Willmore energy satisfies

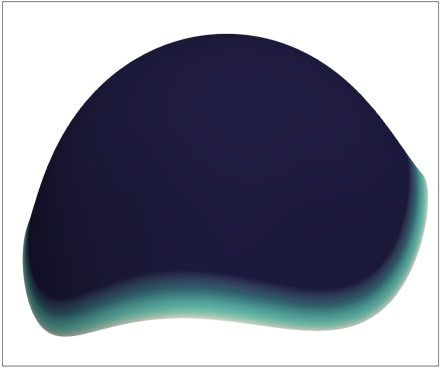
$$\int_{M(t)} \left(\frac{-u_t}{A} \right)^2 + \frac{d}{dt} \int_{M(t)} H^p = 0. \tag{1}$$

- This is GOOD when p is even, since energy is bounded from below.
- When p is odd, stability is highly dependent on initial energy configuration.

Results: graphical p-Willmore flow

3-Willmore evolution of a graphical surface. Initial energy positive (left) and negative (right). Note that a minimal surface is approached in the left case, as suggested by our prior results.





Conjecture for odd p: The p-Willmore flow started from a surface where $\mathcal{W}^p > 0$ remains ≥ 0 for all time.

A flow of closed surfaces

The framework for the closed surface flow is due to Dziuk and Elliott [3]. Consider a parametrization $X_0 : V \subset \mathbb{R}^2 \to \mathbb{R}^3$ of (a portion of) the surface M, and let $u_0 : M \to \mathbb{R}^3$ be identity on M, so $u \circ X = X$.

A variation of M is a smooth function $\varphi : M \to \mathbb{R}^3$ and a 1-parameter family $u(x,t) : M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that $u(x,0) = u_0$ and

$$u(x,t) = u_0(x) + t\varphi(x).$$

Note that this pulls back to a variation $X : V \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$,

$$X(v,t) = X_0(v) + t\Phi(v),$$

where $\Phi = \varphi \circ X$. Note further that (since u is identity on $X(t)$) the time derivatives are related by

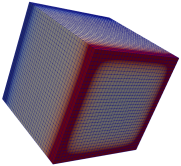
$$\dot{u} = \frac{d}{dt} u(X,t) = \nabla u \cdot \dot{X} + u_t = \dot{X}.$$

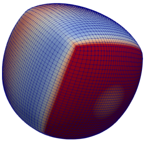
Computational challenges of closed surface flows

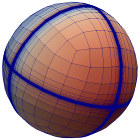
There are notable differences here from the purely theoretical setting:

- Cannot choose a preferential frame in which to calculate derivatives; no natural adaptation (e.g. moving frame) is possible.
- Must consider general variations φ, which may have tangential as well as normal components.
- Must avoid geometric terms that are not easily discretized, such as K and $\nabla_M N$.

Can have very irritating *mesh sliding*:







Later, we will see a fix for this!

Calculating the first variation

Our goal is now to find a weak-form expression for the p-Willmore flow equation,

$$\dot{u} = -\delta \mathcal{W}^p.$$

First, note that the components of the induced metric on M are

$$g_{ij} = \partial_{x_i} X \cdot \partial_{x_j} X = X_i \cdot X_j$$

so that the surface gradient of a function f defined on M can be expressed as

$$(\nabla_M f) \circ X = g^{ij} X_i F_j,$$

where $F = f \circ X$ is the pullback of f through the parametrization X, and $g^{ik} g_{kj} = \delta^i_j$.

The Laplace-Beltrami operator on M is then

$$(\Delta_M f) \circ X = (\nabla_M \cdot \nabla_M f) \circ X = \frac{1}{\sqrt{\det g}} \partial_j \left(\sqrt{\det g}\, g^{ij} F_i \right).$$

Calculating the first variation (2)

Let $Y := \Delta_M u = 2HN$ be the mean curvature vector of $M \subset \mathbb{R}^3$. Then, the p-Willmore functional (modulo a factor of 2^p) can be expressed as

$$\mathcal{W}^p(M) = \int_M (Y \cdot N)^p.$$

It is then relatively straightforward to compute the p-Willmore Euler-Lagrange equation,

$$\frac{p}{2}\Delta_M (Y \cdot N)^{p-1} - p|\nabla_M N|^2 (Y \cdot N)^{p-1} + \frac{1}{2}(Y \cdot N)^{p+1} = 0,$$

for a normal variation of $\mathcal{W}^p$.

Challenges:

- Express this 4^{th} order PDE weakly.
- Include the possibility of tangential motion.
- Suppress derivatives of the vector N.

Possible with some clever rearrangement and a splitting technique applied by G. Dziuk in [4].

The closed surface p-Willmore flow problem

Problem: Closed p-Willmore flow with volume and area constraint

Let $p \geq 2$, $Y = 2HN$, and $W := (Y \cdot N)^{p-2} Y$. Determine a family $M(t)$ of closed surfaces with identity maps $u(X,t)$ such that $M(0)$ has initial volume V_0, initial surface area A_0, and the equation

$$\dot{u} = \delta\left(\mathcal{W}^p + \lambda \mathcal{V} + \mu \mathcal{A}\right),$$

is satisfied for all $t \in (0, T]$ and for some piecewise-constant functions λ, μ.

Equivalently, find functions u, Y, W, λ, μ on $M(t)$ such that the equations

$$\int_M \dot{u} \cdot \varphi + \lambda(\varphi \cdot N) + \mu \nabla_M u : \nabla_M \varphi + \left((1-p)(Y \cdot N)^p - p\nabla_M \cdot W\right)\nabla_M \cdot \varphi$$
$$+ pD(\varphi)\nabla_M u : \nabla_M W - p\nabla_M \varphi : \nabla_M W = 0,$$
$$\int_M Y \cdot \psi + \nabla_M u : \nabla_M \psi = 0,$$
$$\int_M W \cdot \xi - (Y \cdot N)^{p-2} Y \cdot \xi = 0,$$
$$\int_M 1 = A_0,$$
$$\int_M u \cdot N = V_0,$$

are satisfied for all $t \in (0, T]$ and all $\varphi, \psi, \xi \in H_0^1(M(t))$.

How do we implement this? Algorithm:

Let $\tau > 0$ be a fixed step-size and $u^k := u(\cdot, k\tau)$. The p-Willmore flow algorithm proceeds as follows:

1. Given the initial surface position u_h^0, generate the initial curvature data Y_h^0, W_h^0 by solving

$$\int_{M_h^0} Y_h^0 \cdot \psi_h + \nabla_{M_h^0} u_h^0 : \nabla_{M_h^0} \psi_h = 0,$$

$$\int_{M_h^0} W_h^0 \cdot \xi_h - (Y_h^0 \cdot N_h^0)^{p-2} Y_h^0 \cdot \xi_h = 0,$$

for all piecewise-linear test functions φ_h, ψ_h.

Algorithm: p-Willmore flow loop

2. For integer $0 \leq k \leq T/\tau$, flow the surface according to the following procedure:
   1. Solve the (discretized) weak form equations: obtain the positions $\tilde{u}_h^{k+1}$, curvatures $\tilde{Y}_h^{k+1}$ and $\tilde{W}_h^{k+1}$, and Lagrange multipliers λ_h^{k+1} and μ_h^{k+1}.
   2. Minimize conformal distortion of the surface mesh $\tilde{u}_h^{k+1}$, yielding new positions u_h^{k+1}.
   3. Compute the updated curvature information Y_h^{k+1} and W_h^{k+1} from u_h^{k+1}.
3. Repeat step 2 until the desired time T.

Conformal correction step 2.2: idea

To correct mesh sliding at each time step, the goal is to enforce the "Cauchy-Riemann equations" on the tangent bundle TM.

Let $X : V \to \operatorname{Im}\mathbb{H}$ be an immersion of M, and J be a complex structure (rotation operator $J^2 = -\mathrm{Id}_{TV}$) on TV. Then, if $*\alpha = \alpha \circ J$ is the usual Hodge star on forms,

Thm: Kamberov, Pedit, Pinkall [5]

X is conformal iff there is a Gauss map $N : M \to \operatorname{Im}\mathbb{H}$ such that $*dX = N\,dX$.

Note that,

- $N \perp dX(v)$ for all tangent vectors $v \in TV$.
- $v, w \in \operatorname{Im}\mathbb{H} \longrightarrow vw = -v \cdot w + v \times w$.

Conclusion: conformality may be enforced by requiring $*dX(v) = N \times dX(v)$ on a basis for TV!

Conformal correction step 2.2: implementation

Choose x^1, x^2 as coordinates on V, then:

- $\partial_1 := \partial_{x^1}$ and $\partial_2 := \partial_{x^2}$ are a basis for TV.
- $dX(\partial_1) := X_1$ and $dX(\partial_2) := X_2$ are a basis for TM.
- $J\partial_1 = \partial_2$, $J\partial_2 = -\partial_1$.
- $\nabla_{dX(v)} u = \nabla_v X$ on M.

Instead of enforcing conformality explicitly, we minimize an energy functional. First, define

$$\mathcal{CD}_v(u) = \frac{1}{2}\int_M |\nabla_{dX(Jv)} u - N \times \nabla_{dX(v)} u|^2 = \frac{1}{2}\int_M |\nabla_{Jv} X - N \times \nabla_v X|^2.$$

Standard minimization techniques lead to the necessary condition,

$$\delta\,\mathcal{CD} = \int_M \left(\nabla_{dX(Jv)} u - N \times \nabla_{dX(v)} u\right) \cdot \left(\nabla_{dX(Jv)} \varphi - N \times \nabla_{dX(v)} \varphi\right) = 0.$$

Conformal correction step 2.2: implementation (2)

So, choosing the basis $\{X_1, X_2\}$ for TM, it suffices to enforce

$$\int_M (\nabla_{X_2} u - N \times \nabla_{X_1} u) \cdot (\nabla_{X_2} \varphi - N \times \nabla_{X_1} \varphi) + \int_M (\nabla_{X_1} u + N \times \nabla_{X_2} u) \cdot (\nabla_{X_1} \varphi + N \times \nabla_{X_2} \varphi) = 0.$$

Important: To ensure this "reparametrization" does not undo the Willmore flow, we use a Lagrange multiplier ρ to move only on TM.

Specifically, if $\nabla_{M_{h,i}} = \nabla_{M_{h,X_i}}$, we solve for u_h^{k+1}, ρ_h^{k+1} satisfying

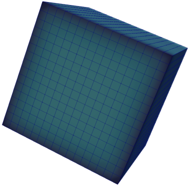
$$\int_{M_h^k} \rho_h^{k+1}(\varphi_h \cdot N_h^k) + (\nabla_{M_{h,2}^k} u_h^{k+1} - N_h^k \times \nabla_{M_{h,1}^k} u_h^{k+1}) \cdot (\nabla_{M_{h,2}^k} \varphi_h - N_h^k \times \nabla_{M_{h,1}^k} \varphi_h) + (\nabla_{M_{h,1}^k} u_h^{k+1} + N_h^k \times \nabla_{M_{h,2}^k} u_h^{k+1}) \cdot (\nabla_{M_{h,1}^k} \varphi_h + N_h^k \times \nabla_{M_{h,2}^k} \varphi_h) = 0,$$

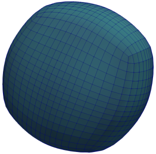
$$\int_{M_h^k} (u_h^{k+1} - \tilde{u}_h^{k+1}) \cdot N_h^k = 0.$$

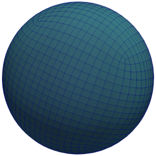
Conformal correction step 2.2: notes

This conformal correction is important because:

- Dramatically improves mesh quality during the p-Willmore flow.
- Keeps simulation from breaking due to mesh degeneration.
- Mitigates the artificial barrier to flow continuation caused by a bad mesh.



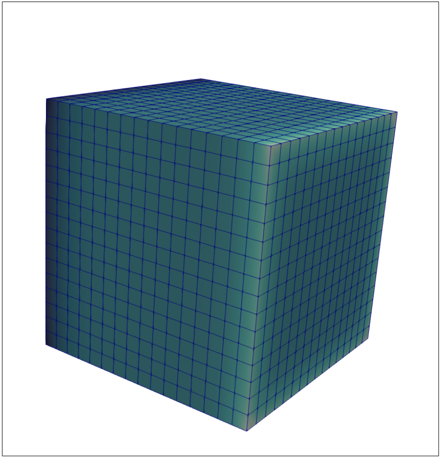


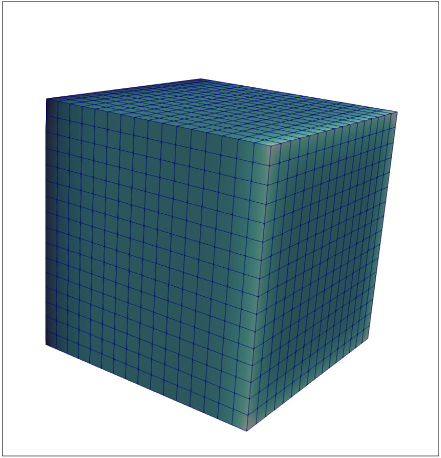


Remark: This procedure can also be extended to triangular meshes with some care.

Results: Willmore vs. $(p > 2)$-Willmore

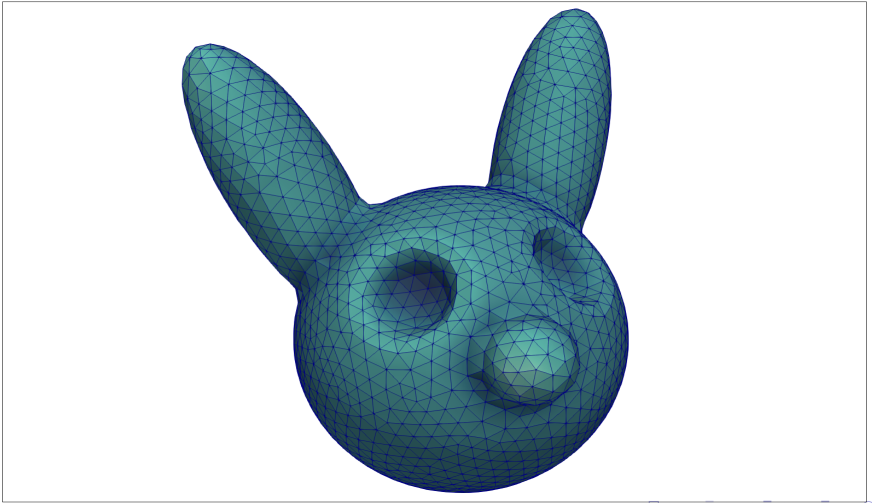
Comparison on a cube: unconstrained Willmore evolution (left) and unconstrained 4-Willmore evolution (right). Note the difference made by conformal invariance.





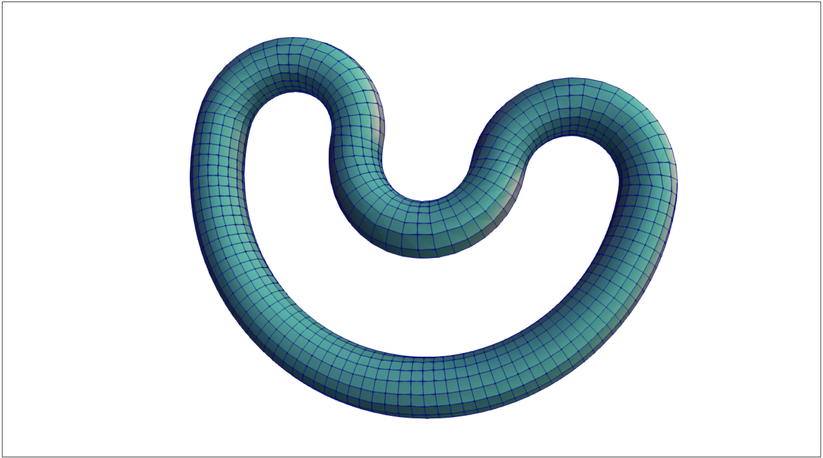
Results: Dog

The 3-Willmore evolution of a genus 0 dog mesh constrained by enclosed volume. Note the initial 3-Willmore energy is positive.



Results: Horseshoe

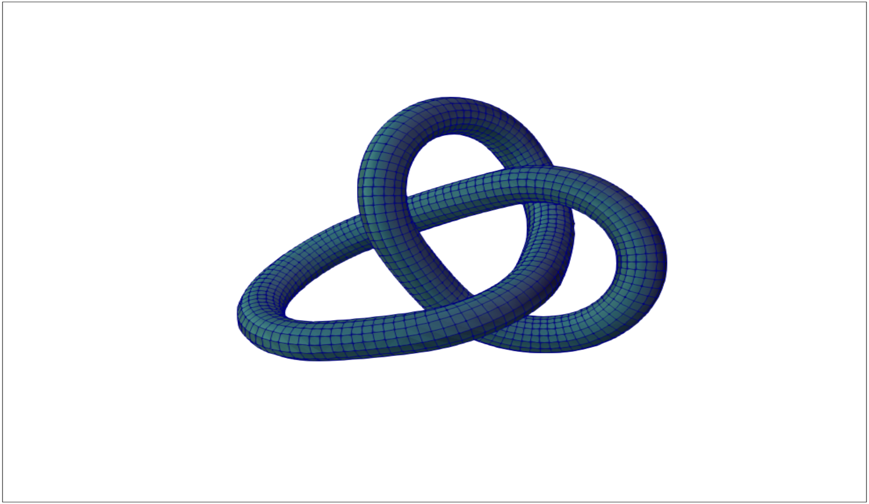
It is not necessary to restrict to genus 0 surfaces. Here is the Willmore flow of a horseshoe surface constrained by volume and surface area.



Note that the poor quality mesh is corrected immediately by the flow.

Results: Knot

The Willmore evolution of a trefoil knot constrained by both surface area and enclosed volume.



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