Approximate and Analytical Solutions of Generalized Lane–Emden–Fowler Equations

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Abstract

• The work deals with a family of *nonlinear differential equations of Lane–Emden–Fowler type*. The original Lane–Emden equation was used to model the thermal behavior of a spherical cloud of gas within the framework of the classical thermodynamics. Slightly modified, it describes phase transitions in critical thermodynamic systems of spherical geometry and other physical phenomena.

• The aim of the current work is to obtain approximate analytical solutions of the regarded equations. The problem is re-formulated in terms of *nonhomogeneous nonlinear Volterra integral equations of the second kind*. The solutions are sought by *He's homotopy perturbation method* and *Picard's method of successive approximations*.

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Overview

Basic equations

The original Lane–Emden and Emden–Fowler equations

• The Lane–Emden equation, see [Lane, 1870], [Emden, 1907] is one of the basic equations in the theory of stellar structure. It has been the focus of many studies. This equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules within the framework of the classical thermodynamics [Chandrasekhar, 1939]. It also describes the variation of density as a function of the radial distance for a polytrope [Novotny, 1972]. The equation has the form

$$\frac{d^2y(x)}{dx^2} + \frac{2}{x}\frac{dy(x)}{dx} + \lambda y^m(x) = 0, \quad \lambda \in \mathbf{R}, \quad m = 0, 1, 2, \ldots \qquad (1)$$

$$y(0) = 1, \quad \left.\frac{dy}{dx}\right|_{x=0} = 0$$

• Exact solutions to Eq. (1) are known only for $m = 0, 1, 5$.

Basic equations

The original Lane–Emden and Emden–Fowler equations

• A two-parameter generalization of the Lane–Emden equation (1), namely

$$\frac{d^2y(x)}{dx^2} + \frac{2}{x}\frac{dy(x)}{dx} + x^{\mu-2}y^m(x) = 0, \quad m = 0, 1, 2, \ldots \tag{2}$$

was introduced and systematically studied by [Fowler, 1930] and is currently known as the Emden–Fowler equation (the Lane–Emden equation corresponds to the particular case $\mu = 2$).

• Two integrable classes of the Emden–Fowler equation (2) with applications in astrophysics and cosmology were found recently, see [Mancas and Rosu, 2018].

• There are many articles in which the Emden–Fowler equation has been solved numerically using different methods and techniques.

Basic equations

A family of generalized Lane–Emden–Fowler equations

- In this work, we consider the generalized Lane–Emden–Fowler equations of the form

$$\frac{d^2y}{dx^2} + \frac{k}{x}\frac{dy}{dx} + x^\nu f(y) = 0, \quad k, \nu \in \mathbf{R} \tag{3}$$

where $f(y)$ is a function of the dependent variable y.

- **Case 1.0.** Let $k = 0$, $\nu = 0$, then Eq. (3) takes the form

$$\frac{d^2y}{dx^2} + f(y) = 0. \tag{4}$$

Obviously, this equation possesses a first integral and hence it is integrable by quadratures. If, moreover, $f(y)$ is a polynomial of third degree, then one can write down the solution of Eq. (4) in explicit analytic form following [Whittaker & Watson, 1927].

Basic equations

A family of generalized Lane–Emden–Fowler equations

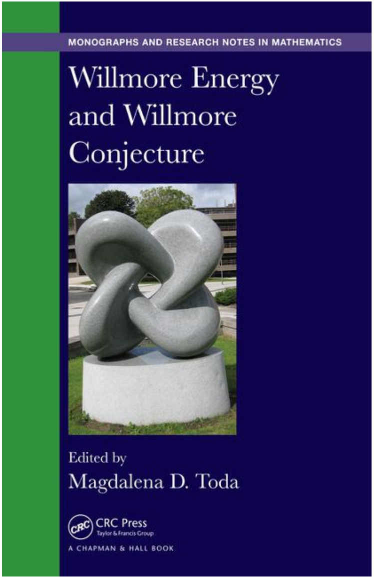
• **Case 1.1.** $k = \nu = 0$, $x := s$, $y := \kappa$, $f(\kappa) = (1/2)\kappa^3 - \mu\kappa - \sigma$, then Eq. (3) takes the form

$$\frac{d^2\kappa}{ds^2} + \frac{1}{2}\kappa^3 - \mu\kappa - \sigma = 0 \tag{5}$$

and describes the cylindrical equilibrium shapes of fluid lipid membranes [Vassilev et al., 2008], elastic rings [Djondjorov et al., 2011], and carbon nanotubes [Mladenov et al., 2013] subjected to uniform hydrostatic pressure. Here s is the arclength of the generating plane curve, κ is its curvature, μ represents the tensile stress or chemical potential, σ represents the hydrostatic pressure.

• Similar equations of this type are studied in Chapter 3 "Analytical Representations of Willmore & Generalized Willmore Surfaces" of [Toda, 2018] "Willmore Energy and Willmore Conjecture".

Willmore Energy and Willmore Conjecture



Chapter 3

Analytical Representations of Willmore and Generalized Willmore Surfaces

Vassil M. Vassilev, Peter A. Djondjorov, Mariana Ts. Hadzhilazova, and Ivailo M. Mladenov

CONTENTS

Abstract

In this chapter we give analytical representations of Willmore and generalized Willmore (Helfrich) surfaces. All cylindrical surfaces of that type are described. Some special families of axially symmetric surfaces providing local extrema to the corresponding functionals are presented. The foregoing surfaces are determined analytically, the components of the respective position vectors being represented either in explicit form or by quadratures in terms of suitable variables. Parametric equations of Dupin cyclides and other Willmore surfaces obtained by inversions of catenoids and Enneper's surface are also presented.

• **Case 1.2.** $k = \nu = 0$, $x := \zeta$, $y := \phi$, $f(\phi) = -2\phi^3 - \tau\phi + (1/2)\eta$, then Eq. (3) becomes

$$\frac{d^2\phi}{d\zeta^2} - 2\phi^3 - \tau\phi + \frac{1}{2}\eta = 0 \tag{6}$$

and describes, within the standard ϕ^4 Ginzburg-Landau theory of phase transitions, the order parameter ϕ at the position $\zeta \in (0, L)$ perpendicular to the bounding planes of a thin film of thickness L; τ and η represent the temperature and ordering field.

• This equation subject to several tips of boundary conditions, e.g., $(+,+)$, $(+,-)$, etc., has been studied in a series of recent works by Dantchev, Vassilev and Djondjorov where the interested reader can find exact analytical solution for the respective order parameter profiles, Casimir forces, local and total susceptibilities, etc.

Some papers

Journal of Statistical Mechanics: Theory and Experiment
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Exact results for the temperature-field behavior of the Ginzburg–Landau Ising type mean-field model

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Analytical results for the Casimir force in a Ginzburg–Landau type model of a film with strongly adsorbing competing walls

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Exact solution for the order parameter profiles and the Casimir force in ^{4}He superfluid films in an effective field theory

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- **Case 2.1.** $k = d - 1$, $\nu = 0$, $x := r$, $y := \phi$, $f(\phi)$ as in Case 1.2, then Eq. (3) reads

$$\frac{d^2\phi}{dr^2} + \frac{d-1}{r}\frac{d\phi}{dr} - 2\phi^3 - \tau\phi + \frac{1}{2}\eta = 0 \tag{7}$$

and describes, within the foregoing ϕ^4 Ginzburg-Landau theory cast in spherical co-ordinates (r, θ, φ), the order parameter ϕ, which is assumed to depend only on the radial co-ordinate r, at the position $r > 0$ normal to the bounding spheres of a film of sphere geometry; as before, τ and η represent the temperature and ordering field.

- Within the framework of the regarded theory, d is the dimension of a hypersphere, which is of one dimension less than that of the space itself. Actually, the most interesting cases are $d = 3$ and $d = 4$.

Basic equations

Representation via Volterra integral equations

- The generalized Lane–Emden–Fowler equation (3) , i.e.

$$\frac{d^2y}{dx^2} + \frac{k}{x}\frac{dy}{dx} + x^{\nu} f(y) = 0, \quad k, \nu \in \mathbf{R} \tag{8}$$

is equivalent, except for the case $k = 1$, to the following nonlinear nonhomogeneous Volterra integral equation of the second kind

$$y(x) = a + b\, x^{1-k} + \frac{1}{k-1}\int_c^x \left(x^{1-k} s^{k-\nu} - s^{\nu+1}\right) f\left(y\left(s\right)\right) ds, \tag{9}$$

where $a, b, c \in \mathbf{R}$ are constants to be specified by the imposed initial and/or boundary conditions using also the expression

$$\frac{dy(x)}{dx} = b\left(1-k\right) x^{-k} - x^{-k}\int_c^x s^{k+\nu} f\left(y\left(s\right)\right) ds \tag{10}$$

for the derivative of the sought function $y(x)$ following from (9).

- The integral equation (9) can be written in the form

$$\mathcal{L}[y] = 0,$$

$$\mathcal{L}[y] = y(x) - a - bx^{1-k} - \frac{1}{k-1}\int_c^x \left(x^{1-k}s^{k-\nu} - s^{\nu+1}\right) f\left(y\left(s\right)\right) ds$$

and solved using He's homotopy perturbation method [He, 1999].

- In general, HPM assumes to define a homotopy function/operator

$$H(y,p) = (1-p)\mathcal{F}[y] + p\mathcal{L}[y] = 0, \qquad p \in [0,1], \tag{11}$$

where $\mathcal{F}[y] = H(y,0)$ is a function/operator with known solution. Then, increasing monotonically the parameter p from zero to one, to deform continuously a solution $y_0(x)$ of the equation $\mathcal{F}[y] = 0$ to a solution of the equation $\mathcal{L}[y] = H(y,1) = 0$.

• Within the HPM [He, 1999] the solution of equation $H(y,p)=0$ is sought as a power series in the homotopy parameter p, i.e.

$$y(x,p) = y_0(x) + py_1(x) + p^2 y_2(x) + \cdots \,, \tag{12}$$

where, as assumed above, $y_0(x)$ is a known solution of the equation $\mathcal{F}[y]=0$. The functions $y_1(x), y_2(x), \ldots$ are to be determined in the following way.

▶ Step 1. Choose the function/operator $\mathcal{F}[y]$ – the "starting point".
▶ Step 2. Substitute expression (12) into Eq. (11) for the homotopy function.

▶ Step 3. Expand (if necessary) the nonlinearities as power series in the homotopy parameter p about $p=0$.

▶ Step 4. Set the coefficients at p^n, $n=0,1,2,\ldots$ to zero.

- **Case 2.1.1.** $k = 3$, $\nu = 0$, $x := r$, $y := \phi$, $f(\phi)$ – as in Case 1.2, then Eq. (3) reads

$$\frac{d^2\phi}{dr^2} + \frac{3}{r}\frac{d\phi}{dr} - 2\phi^3 - \tau\phi + \frac{1}{2}\eta = 0 \tag{13}$$

and can be represented by the following Volterra integral equation of the second kind

$$\mathcal{L}[\phi] = \phi(r) - a - b\,r^{-2} - \frac{1}{2}\int_c^r s\left(r^{-2}s^2 - 1\right) f\left(\phi\left(s\right)\right) ds = 0. \tag{14}$$

- We shall assume that the initial conditions are of the form

$$\phi(0) = a, \quad \left.\frac{d\phi}{dr}\right|_{r=0} = 0 \tag{15}$$

and, accordingly, $b = c = 0$.

Application of homotopy perturbation method

An example from ϕ^4 Ginzburg-Landau theory of phase transitions

- We choose the "starting point" to be $\mathcal{F}[\phi] = \phi(r) - a$, and use the following homotopy operator

$$H(\phi, p) = (1-p)\mathcal{F}[\phi] + p\mathcal{L}[\phi] = 0. \qquad p \in [0,1], \tag{16}$$

- The solution of equation $H(\phi, p) = 0$ is sought in the form

$$\phi(r,p) = \phi_0(r) + p\,\phi_1(r) + p^2\phi_2(r) + \cdots , \tag{17}$$

where $\phi_0(r) = a$ since this is the solution of equation $\mathcal{F}[\phi] = 0$.

- Substituting (17) into equation (16) and equating the coefficients at p^n, $n = 0, 1, 2, \ldots$ to zero, we have obtained the functions $\phi_1(r), \ldots, \phi_5(r)$ performing symbolic computation with Wolfram *Mathematica*®9.
- The computation took 22 sec. on a PS of i7 CPU, 32 GB RAM.

Application of Picard's method

- The solution of the regarded Volterra integral equation (14), i.e.

$$\phi(r) = a + \frac{1}{2}\int_0^r s\left(r^{-2}s^2 - 1\right) f\left(\phi\left(s\right)\right) ds = 0. \tag{18}$$

can be sought using Picard's method of successive approximations (see, e.g., [Tricomi, 1957]) as the limit of a sequence $\{\varphi_j(r)\}$ of functions whose first element is the given function $\varphi_0(r) = a$, the other elements being calculated by the recurrence formula

$$\varphi_{j+1}(r) = a + \frac{1}{2}\int_0^r s\left(r^{-2}s^2 - 1\right) f\left(\varphi_j\left(s\right)\right) ds = 0. \tag{19}$$

- This sequence converges under some reasonable assumptions about the properties of the function $f\left(\phi\left(s\right)\right)$.

- Performing symbolic computation with Wolfram *Mathematica*®9 again on the same PS, we have obtained (after almost 5 hours) the functions $\varphi_1\left(r\right), \ldots, \varphi_5\left(r\right)$.

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