Soliton surface for the (1+1)-dimensional Shrödinger-Maxwell-Bloch equation

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1. The (1+1)-dimensional Schrödinger-Maxwell-Bloch equation

Optical soliton propagation in fibres with resonant and erbium-doped systems is governed by the coupled systems of the SMBE.
The (1+1)-dimensional Schrödinger-Maxwell-Bloch equation (SMBE) has form

$$iq_t + q_{xx} + 2|q|^2 q - 2ip = 0, \tag{1}$$

$$p_x - 2i\omega_0 p - 2\eta q = 0, \tag{2}$$

$$\eta_x + qp^* + q^* p = 0, \tag{3}$$

where q, p are complex variables functions, and η is a real variable function, ω_0 is a real constant. This (1+1)-dimensional SMBE is integrable are given by ISP.

1.1 Lax representation of the (1+1)-dimensional SMBE

The corresponding Lax representation of equations (1)-(3) reads as

$$\Psi_x = U\Psi, \quad (4)$$

$$\Psi_t = V\Psi, \quad (5)$$

where $\Psi = (\Psi_1, \Psi_2)^T$ is vector eigenfunction and U, V are matrices, depending on the complex eigenvalue parameter λ:

$$U = \begin{pmatrix} -i\lambda & q \\ -q^* & i\lambda \end{pmatrix} \equiv -i\lambda\sigma_3 + U_0, \quad (6)$$

$$V = i\begin{pmatrix} i\lambda^2 & i\lambda q \\ i\lambda q^* & -i\lambda^2 \end{pmatrix} + \begin{pmatrix} |q|^2 & q_x \\ q_x^* & -|q|^2 \end{pmatrix} + \frac{i}{\lambda+\omega_0}\begin{pmatrix} \eta & -p \\ -\bar{p} & -\eta \end{pmatrix} \equiv$$

$$\equiv i\lambda^2 V_2 + i\lambda V_1 + iV_0 + \frac{i}{\lambda+\omega_0}V_{-1}. \quad (7)$$

1.2.The Darboux transformation.

The Darboux transformation is very efficient for construction of soliton solutions. Based on the Darboux transformation for AKNS system, we consider the following transformation of the SMBE

$$\Psi^{[1]} \quad = \quad T\Psi = (\lambda I - M)\,\Psi, \tag{8}$$

where $\Psi^{[1]}$, Ψ are eigenfunctions, T is the Darboux matrix, M and I matrices have the form

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Substituting (8) into the Lax pair (4)-(5) we obtain expressions for $\Psi^{[1]}$

$$\psi_x^{[1]} = U^{[1]}\psi^{[1]}, \tag{9}$$
$$\psi_t^{[1]} = V^{[1]}\psi^{[1]}, \tag{10}$$

where $U^{[1]}$ and $V^{[1]}$ depend on $q^{[1]}, p^{[1]}, \eta^{[1]}$ and λ, respectively. In order to hold the equations (9) and (10), T is the Darboux matrix and must satisfy the next equalities

$$T_x + TU = U^{[1]}T, \tag{11}$$
$$T_t + TV = V^{[1]}T. \tag{12}$$

Finally, we have DT of the SMBE:

$$q^{[1]} = q + 2m_{12}, \tag{13}$$

$$\eta^{[1]} = \frac{1}{\square}\left[|i\omega_0 + m_{11}|^2 - |m_{12}|^2\right)\eta - pm_{12}^*(i\omega_0 + m_{11}) - \\ -p^* m_{12}(i\omega_0 + m_{11}^*)\Big], \tag{14}$$

$$p^{[1]} = \frac{1}{\square}\left[p(i\omega_0 + m_{11})^2 - p^* m_{12}^2 + 2\eta m_{12}(i\omega_0 + m_{11})\right], \tag{15}$$

Here symbol $\square$ has form

$$\square = det(M + i\omega_0 I) = -\omega_0^2 + i\omega_0(m_{11} + m_{11}^*) + |m_{11}|^2 + |m_{12}|^2. \tag{16}$$

Having the explicit form the DT (13)-(15) of the SMBE, we can construct exact solutions. To get one-soliton solutions we assume trivial seed solutions as

$$q = p = 0, \quad \eta = 1. \tag{17}$$

Then the corresponding associated linear system takes the form

$$\Psi_{1x} = -i\lambda\Psi_1, \tag{18}$$

$$\Psi_{2x} = i\lambda\Psi_2, \tag{19}$$

$$\Psi_{1t} = \left(-2i\lambda^2 + \frac{i}{\lambda + \omega_0}\right)\Psi_1, \tag{20}$$

$$\Psi_{2t} = \left(2i\lambda^2 - \frac{i}{\lambda + \omega_0}\right)\Psi_2, \tag{21}$$

This system admits the following exact solutions

$$\psi_{11} = \exp\left[\lambda_1 x + \left(i\lambda_1^2 + \frac{1}{\lambda_1 + i\omega_0}\right) t + \frac{x_0 + iy_0}{2}\right], \qquad (22)$$

$$\psi_{21} = \exp\left[-\lambda_1 x - \left(i\lambda_1^2 + \frac{1}{\lambda_1 + i\omega_0}\right) t - \frac{x_0 + iy_0}{2} + iz\right] \qquad (23)$$

and x_0, y_0, z and ω_0 are real constants. Here $\lambda_1 = a_1 + ib_1$ $(a_1, b_1 \in R)$.

Then the one-soliton solution of (1+1) dimensional SMBE is derived as

$$q^{[1]} = 2a_1 sech\left[\tilde{x}\right] \exp\left[i\tilde{y} - iz\right],$$

$$p^{[1]} = 2a_1 \frac{sech^2[\tilde{x}]\left(a_1 \sinh[\tilde{x}] + i\left(b_1 + \omega_0\right)\cosh[\tilde{x}]\right)}{a_1^2 + \left(b_1 + \omega_0\right)^2} \exp i[\tilde{y} - z],$$

$$\eta^{[1]} = 1 - 2\frac{a_1^2 sech^2[\tilde{x}]}{a_1^2 + \left(b_1 + \omega_0\right)^2},$$

where

$$\tilde{x} = 2a_1 x + \left(-4a_1 b_1 + \frac{2a_1}{a_1^2 + \left(b_1 + \omega_0\right)^2}\right) t + x_0,$$

$$\tilde{y} = 2b_1 x + \left(2\left(a_1^2 - b_1^2\right) - \frac{2(b_1 + \omega_0)}{a_1^2 + \left(b_1 + \omega_0\right)^2}\right) t + y_0.$$

2. The fundamental form.

2.1.The first fundamental form for the (1+1)- dimensional SMBE.

In general, the first and second fundamental forms are

$$I = g_{ij} dx^i dx^j, \tag{24}$$

$$II = b_{ij} dx^i dx^j, \tag{25}$$

here g_{ij}, b_{ij} are matrices

$$g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad b_{ij} = \begin{pmatrix} e & f \\ f & d \end{pmatrix} \tag{26}$$

Position vector is

$$\vec{r} = (r_1, r_2, r_3), \tag{27}$$

and normal to the surface is

$$\vec{n} = (n_1, n_2, n_3), \quad \vec{n}^2 = 1. \tag{28}$$

Using Sym-Tafel formula

$$r = \Phi^{-1}\Phi_\lambda, \tag{29}$$

we can get position matrices

$$r_x = \Phi^{-1}U_\lambda\Phi, \quad r_t = \Phi^{-1}V_\lambda\Phi. \tag{30}$$

The fundamental forms can be presented through position and normal vectors:

$$I = \vec{d}r \cdot \vec{d}r = \vec{r}_x^2 dx^2 + 2\vec{r}_x \vec{r}_t dxdt + \vec{r}_t^2 dt^2, \tag{31}$$

or

$$I = Edx^2 + 2Fdxdt + Gdt^2, \tag{32}$$

$$II = -\vec{d}n \cdot \vec{d}r = (\vec{n} \cdot \vec{r}_{xx})\, dx^2 + 2\,(\vec{n} \cdot \vec{r}_{xt})\, dxdt + (\vec{n} \cdot \vec{r}_{tt})\, dt^2, \tag{33}$$

or

$$II = edx^2 + 2fdxdt + gdt^2. \tag{34}$$

Relations between derivations of vector and matrix form of r with respect to x and t:

$$\vec{r}_x^2 = \frac{1}{2} tr\left(r_x^2\right), \quad (35)$$

$$\vec{r}_t^2 = \frac{1}{2} tr\left(r_t^2\right), \quad (36)$$

$$\vec{r}_x \vec{r}_t = \frac{1}{2} tr\left(r_x r_t\right). \quad (37)$$

Now, we obtain the necessary quantities

$$r_x^2 = \Phi^{-1} U_\lambda^2 \Phi, \quad (38)$$

$$r_t^2 = \Phi^{-1} V_\lambda^2 \Phi, \quad (39)$$

$$r_x r_t = \Phi^{-1\lambda} U_\lambda V_\lambda \Phi. \quad (40)$$

Now we find traces of (38) - (40)

$$tr(r_x^2) = -2, \tag{41}$$

$$tr(r_t^2) = -2\left(16\lambda^2 + 4\left|q\right|^2 + \frac{\eta^2 + \left|p\right|^2}{\left(\lambda + \omega_0\right)^4} + \right.$$
$$\left. + \frac{2i}{\left(\lambda + \omega_0\right)^2}\left(\bar{q}p - q\bar{p}\right) + \frac{8\lambda\eta}{\left(\lambda + \omega_0\right)^2}\right), \tag{42}$$

$$tr\left(U_\lambda V_\lambda\right) = -2\left(4\lambda + \frac{\eta}{\left(\lambda + \omega_0\right)^2}\right). \tag{43}$$

Finally, we get the first fundamental form for (1+1)-dimentional SMBE:

$$I = dx^2 + 2\left(16\lambda^2 + 4|q|^2 + \frac{\eta^2 + |p|^2}{(\lambda+\omega_0)^4} + \right.$$
$$\left. +2\frac{i(\bar{q}p - q\bar{p}) + 4\lambda\eta}{(\lambda+\omega_0)^2}\right) dxdt + \left(4\lambda + \frac{\eta}{(\lambda+\omega_0)^2}\right) dt^2, \quad (44)$$

2.2.The second fundamental form for the (1+1)- dimensional SMBE.

Using Sym-Tafel formula (29) we can find the next

$$r_{xx} = \Phi^{-1}\left[U_\lambda, U\right]\Phi, \quad (45)$$

$$r_{xt} = \Phi^{-1}\left[U_\lambda, V\right]\Phi, \quad (46)$$

$$r_{tt} = \Phi^{-1}\left[V_\lambda, V\right]\Phi. \quad (47)$$

But a normal to surface can be calculated by formula

$$n = \frac{\Phi^{-1}\left[U_\lambda, V_\lambda\right]\Phi}{\sqrt{\frac{1}{2}tr\left(\left[U_\lambda, V_\lambda\right]^2\right)}} \quad (48)$$

Relations between derivations of vector and matrix form of r with respect to x and t:

$$\vec{n} \cdot \vec{r}_{xx} = \frac{1}{2} tr\left(n \cdot r_{xx}\right), \tag{49}$$

$$\vec{n} \cdot \vec{r}_{xt} = \frac{1}{2} tr\left(n \cdot r_{xt}\right), \tag{50}$$

$$\vec{n} \cdot \vec{r}_{tt} = \frac{1}{2} tr\left(n \cdot r_{tt}\right). \tag{51}$$

Traces are determined by the next form:

$$tr\left(n\cdot r_{xx}\right) = \frac{tr\left(\left[U_\lambda, V_\lambda\right]\left[U_\lambda, U\right]\right)}{\sqrt{\frac{1}{2}tr\left(\left[U_\lambda, V_\lambda\right]^2\right)}}, \quad (52)$$

$$tr\left(n\cdot r_{xt}\right) = \frac{tr\left(\left[U_\lambda, V_\lambda\right]\left[U_\lambda, V\right]\right)}{\sqrt{\frac{1}{2}tr\left(\left[U_\lambda, V_\lambda\right]^2\right)}}, \quad (53)$$

$$tr\left(n\cdot r_{tt}\right) = \frac{tr\left(\left[U_\lambda, V_\lambda\right]\left[V_\lambda, V\right]\right)}{\sqrt{\frac{1}{2}tr\left(\left[U_\lambda, V_\lambda\right]^2\right)}}. \quad (54)$$

The second fundamental form (33) takes the form

$$II = -\frac{1}{2\mu}\left\{\alpha dx^2 + 2\beta dxdt + \gamma dt^2\right\} \tag{55}$$

where

$$\alpha = tr\left(\left[U_\lambda, U\right]\left[U_\lambda, V_\lambda\right]\right), \tag{56}$$

$$\beta = tr\left(\left[U_\lambda, V\right]\left[U_\lambda, V_\lambda\right]\right), \tag{57}$$

$$\gamma = tr\left(\left[V_\lambda, V\right]\left[U_\lambda, V_\lambda\right]\right), \tag{58}$$

$$\mu = \sqrt{\frac{1}{2} tr\left(\left[U_\lambda, V_\lambda\right]^2\right)}, \tag{59}$$

3. Area of surface for the (1+1)- dimensional SMBE.

Surface's area is given in the form

$$S = \iint \sqrt{g}\, dxdt = \iint \left| \vec{r}_x \wedge \vec{r}_t \right| dxdt, \qquad (60)$$

where

$$g = \det\left(g_{ij}\right) = \det \begin{pmatrix} \vec{r}_x^{\,2} & \vec{r}_x \cdot \vec{r}_t \\ \vec{r}_x \cdot \vec{r}_t & \vec{r}_t^{\,2} \end{pmatrix}. \qquad (61)$$

Taking into account (45), we get

$$S = \iint \sqrt{\frac{1}{2} tr \left\{ [U_\lambda, U] \right\}^2} dxdt \tag{62}$$

So we can write the surface area using Lax pairs.

$$[U_\lambda, U] = -i\lambda[\sigma_3, U_0] = -2i \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \tag{63}$$

$$[U_\lambda, U]^2 = 4 \begin{pmatrix} |q|^2 & 0 \\ 0 & |q|^2 \end{pmatrix}, \tag{64}$$

$$tr \left([U_\lambda, U]^2 \right) = 8|q|^2. \tag{65}$$

Finally, we get

$$S = 2 \iint \sqrt{|q|^2} dxdt \tag{66}$$

4. Christoffel symbols

The Gauss equations associated with a surface are

$$\vec{r}_{xx} = \Gamma^1_{11}\vec{r}_x + \Gamma^2_{11}\vec{r}_t + e\vec{n}, \tag{67}$$

$$\vec{r}_{xt} = \Gamma^1_{12}\vec{r}_x + \Gamma^2_{12}\vec{r}_t + f\vec{n}, \tag{68}$$

$$\vec{r}_{tt} = \Gamma^1_{22}\vec{r}_x + \Gamma^2_{22}\vec{r}_t + g\vec{n}. \tag{69}$$

The Γ^i_{jk} in (71)-(73) are the usual Christoffel symbols given by the relations

$$\Gamma^i_{jk} = \frac{g^{il}}{2}\left(g_{jl,k} + g_{kl,j} - g_{jk,l}\right) \tag{70}$$

The Gauss equations associated with a surface are

$$\Gamma_{11}^{1}=-\frac{2\eta_{x}}{\Delta\left(\lambda+\omega_{0}\right)^{2}}\left(4\lambda+\frac{\eta}{\left(\lambda+\omega_{0}\right)^{2}}\right), \tag{71}$$

$$\Gamma_{11}^{2}=\frac{2\eta_{x}}{\Delta\left(\lambda+\omega_{0}\right)^{2}}, \tag{72}$$

$$\Gamma_{12}^{1}=-\frac{1}{2\Delta}\left(4\lambda+\frac{\eta}{\left(\lambda+\omega_{0}\right)^{2}}\right)\left(4|q|_{x}^{2}+\frac{2\eta\eta_{x}+|p|_{x}^{2}}{\left(\lambda+\omega_{0}\right)^{4}}+\right.$$
$$\left.+\frac{2i\left(\bar{q}p-q\bar{p}\right)_{x}+8\lambda\eta_{x}}{\left(\lambda+\omega_{0}\right)^{2}}\right) \tag{73}$$

$$\Gamma_{12}^{2} = \frac{1}{2\Delta}\left[4|q|_x^2 + \frac{2\eta\eta_x + |p|_x^2}{(\lambda+\omega_0)^4} + \frac{2i\left(\bar{q}p - q\bar{p}\right)_x + 8\lambda\eta_x}{(\lambda+\omega_0)^2}\right] \quad (74)$$

$$\Gamma_{22}^{1} = \frac{1}{2\Delta}\left(16\lambda^2 + 4\left|q\right|^2 + \frac{\eta^2 + \left|p\right|^2}{(\lambda+\omega_0)^4} + 2\frac{i\left(\bar{q}p - q\bar{p}\right) + 4\lambda\eta}{(\lambda+\omega_0)^2}\right)\cdot$$

$$\cdot\left(\frac{4\eta_t}{(\lambda+\omega_0)^2} - 4|q|_x^2 - \frac{2\eta\eta_x + |p|_x^2}{(\lambda+\omega_0)^4} - \frac{2i\left(\bar{q}p - q\bar{p}\right)_x + 8\lambda\eta_x}{(\lambda+\omega_0)^2}\right) -$$

$$-\left(4\lambda + \frac{\eta}{(\lambda+\omega_0)^2}\right)\left(4|q|_t^2 + \frac{2\eta\eta_t + |p|_t^2}{(\lambda+\omega_0)^4} + \frac{2i\left(\bar{q}p - q\bar{p}\right)_t + 8\lambda\eta_t}{(\lambda+\omega_0)^2}\right) \quad (75)$$

$$\Gamma_{22}^{2}=\frac{1}{2\Delta}\left[\left(4\lambda+\frac{\eta}{(\lambda+\omega_0)^2}\right)\left(4|q|_x^2+\frac{2\eta\eta_x+|p|_x^2}{(\lambda+\omega_0)^4}+\right.\right.$$

$$\left.+\frac{2i\left(\bar{q}p-q\bar{p}\right)_x+8\lambda\eta_x}{(\lambda+\omega_0)^2}-\frac{4\eta_t}{(\lambda+\omega_0)^2}\right)+4|q|_t^2+\frac{2\eta\eta_t+|p|_t^2}{(\lambda+\omega_0)^4}+$$

$$\left.+\frac{2i\left(\bar{q}p-q\bar{p}\right)_t+8\lambda\eta_t}{(\lambda+\omega_0)^2}\right] \tag{76}$$

where

$$\Delta=EG-F^2=\left(2\bar{q}-\frac{i\bar{p}}{(\lambda+\omega_0)^2}\right)\left(2q+\frac{ip}{(\lambda+\omega_0)^2}\right) \tag{77}$$

5. Conclusions and (some) open problems.

1. Lax pair is presented.
2. Darboux is constructed.
3. Soliton solutions are found.
4. Fundamental forms are obtained.
5. Area is found.
6. Soliton surface will be obtained.

Thank you for attention!