Chapter 5
Equations of Equilibrium States of Membranes

Abstract The energy of the bending of the membranes within the Canham model is directly justified as an extension of the case of bending of the elastic strips known as Euler's Elasticas. Later, this model was elaborated upon by Helfrich & Deuling in the form of a non-linear system of two equations for the curvatures of the axially-symmetric membranes. Finally, Ou-Yang and Helfrich introduced the model which is currently seen to be most adequate for the description of membrane configurations. Since we are primarily interested in analytical solutions, it is quite natural that the equation by Ou-Yang and Helfrich be examined for the presence of symmetries, because they are in direct connection with solutions. When this is done, it becomes clear that the most general group of symmetries of the shape equation of the form coincides with the group of Euclidean motions in the real three-dimensional space. Among the generators of this group are those of rotations and translations, which hint about the existence of the analytical solutions discussed in the present and the subsequent chapters.

5.1 Canham Model

The evaluation of the stored elastic energy in a thin plate bent into two perpendicular planes is very complicated. However, if the shear stresses are zero, it would appear that the problem reduces to the integral of the sum of the squares of the two curvatures (Fig. 5.1).

It has been necessary to deviate considerably from the equations of structural engineering in order to accommodate the abundant biological observations, e.g., the red blood cells (RBC) can adopt so many shapes without hemolysis and can change from the crenated form into a biconcave shape.

To solve the problem concretely with an RBC shape, Canham (1970) had considered the membrane's elastic energy of bending in the form

$$U = \frac{D}{2}\int(\kappa_1^2 + \kappa_2^2)\mathrm{d}\mathcal{A}, \tag{5.1}$$

where D is the bending rigidity, and κ_1 and κ_2 are the principle curvatures of the surface based on the model described in the next section.

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