Chapter 6
Exact Solutions and Applications

Abstract This chapter considers the construction of the few known and some new explicit solutions to the shape equation. For example, a review of the available experimental facts concerning nerve fibers suggests that their form can be adequately modelled by Delaunay unduloids. We present parameterizations of these surfaces in terms of the geometrical parameters involved in the model. This allows for direct expression of all geometric characteristics to the membranes, like the length of any separate extension, their volume and surface area via explicit formulas. This in turn allows for the direct examination of sensitivity of these characteristics on the amplitude oscillations of the parameters and animation of the morphological changes. Note that this is either impossible or very difficult to achieve by means of numerical analysis. The same applies to the modelling of the results in Cole's experiments, which can be described in terms of Delaunay's nodoids. Besides analytical formulas, we consider the geometry of the egg of the sea urchin under bilateral deformation between two plates of the model and determine the surface tension of the membrane of the egg as a direct consequence of analytical results. Another instance when Delaunay's nodoids appear quite naturally is in the modelling of the membrane fusion. According to the commonly-accepted assumption, this process involves an hour-glass-shaped local contact between two monolayers of opposing membranes, an intermediate structure which is called a stalk. The shape of the stalk is considered to be an axisymmetrical figure of revolution in the three-dimensional space with a planar geometry in the initial configuration. The total energy of the stalk is evaluated from the assumption that the stalk is a nodoid with constant curvature. The conclusions of the group analysis in the previous chapter are used to distinguish two classes of analytical solution of the shape equation, the translationally-invariant and axially-symmetric, to be precise. In the first case, the Ou-Yang equation is reduced to the equation for a so-called generalized elastica, whose solutions are fully described and illustrated with graphics. The second class of axially-symmetric membranes is the family of Delaunay-like surfaces, but with non-constant mean curvatures. These surfaces constitute the first examples of a surfaces with periodic mean curvatures.

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