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STRAIGHT AND CIRCULAR MOTIONS OF A PARTICLE IN THE FIELD OF TWO FIXED ATTRACTORS

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Abstract. The two centre problem is studied both in the phase plane and in spacetime, assuming first a trajectory collinear, and, as a second case, a circular one through the newtonian attractors, finding a saddle equilibrium. For the latter problem a probably new differential equation is met and solved. Time is then obtained in both cases through elliptic integrals of all kinds and Jacobian functions.

1. Introduction

This problem consists of computing the motion of a test particle in the field of two fixed centres of newtonian attraction. It was first considered by Euler in 1760, who showed its integrability (see [7] for the early history of this problem). Nowadays, the system plays an important role both in macro and microphysics. In the past it represents a body moving under the attraction of two fixed stars. Passing to relativistic implications, Contopoulos *et al.*, [3], [4], have discovered, through numerical experiments, that in contrast with the classic two-centre problem, whose dynamics is completely integrable, relativistic motion of two black-holes in spacetime exhibits *chaotic* behavior. In the latter, the system is the simplest model of a diatomic molecule, since Pauli had applied it to the hydrogen molecular ion H_2^+ in his doctoral thesis, 1922, well before the birth of wave mechanics. Anyway, the assumption that their nuclei are fixed is known as “Born-Oppenheimer approximation” whose paper *Zur Quantentheorie der Molekeln*, 1927, describes the separation of electronic motion, nuclear vibrations, and molecular rotation. Such approximation is ubiquitous in quantum chemical calculations, the test particles being electrons which are assumed to “feel” the Coulomb attractive potential V of the nuclei clamped at certain space positions. Generalizing the attraction law to $V = ar^{2n}$, where r is a distance and n a real number, it has recently proved, [6], that a two fixed attractors problem is integrable when: