JGSP **16** (2009) 23–37

JOURNAL OF
Geometry and Symmetry in Physics

CHARACTERIZATION AND COMPUTATION OF CLOSED GEODESICS ON TOROÏDAL SURFACES

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Communicated by John Oprea

Abstract. The aim of the present study is to characterize and compute closed geodesics on toroïdal surfaces. We show that a closed geodesic must make a number of rotations about the equatorial part (k rotations) and the axis of revolution (k' rotations) of the surface. We give the relation that exists between the numbers k and k', and the Clairaut's constant C corresponding to the geodesic. Moreover, we prove that the numbers k and k' are relatively prime. We validate our findings by constructing closed geodesics on some examples of toroïdal surfaces using MAPLE. Finally, using experimental data on cardiac fiber direction, we show that fibers run as geodesics in the left ventricle whose geometrical shape looks like a toroïdal surface.

1. Introduction

Geodesics are of high importance due to their wide applications in many fields such as topography, biology, etc. For instance, according to Streeter [14], in the equatorial part of the left ventricle free wall, fibers are organized into toroïdal surfaces on which they run as geodesics.

From a mathematical point of view, a geodesic on a parameterized surface is a solution to a nonlinear system of two second order ordinary differential equations. Consequently, this representation leads to the local existence of a geodesic starting from a given point and tangent to a given vector belonging to the tangent plane at the point, see Berger and Gostiaux [3]. On the other hand, global existence results are available in the case of closed surfaces. Indeed, there is a global result that guaranties that every local geodesic can be extended to a geodesic defined over $\mathbb{R}$, see Schwartz [13]. Moreover, another result states that every two points of a closed surface can always be connected by at least a geodesic with minimal length (particular case of the Hopf-Rinow theorem, [6]). Closed geodesics in particular, have been studied intensively, however, their existence is not straightforward. We