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THE GAUSS MAP OF MINIMAL GRAPHS IN THE HEISENBERG GROUP

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Abstract. In this paper we study some geometric properties of surfaces in the Heisenberg group, $\mathcal{H}_3$. We obtain, using the Gauss map for Lie groups, a partial classification of minimal graphs in $\mathcal{H}_3$.

1. Introduction

The classical Heisenberg group, $\mathcal{H}_3$, is the group of 3×3 matrices of the form

$$\begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \qquad r, t, s \in \mathbb{R}. \tag{1}$$

This group is a two-step nilpotent (or quasi-abelian) Lie group, which is the nearest condition to be abelian. Endowed with a left invariant metric g, the isometry group of $(\mathcal{H}_3, g)$ is four-dimensional. It is known that there is no three-dimensional Riemannian manifold with isometry group of dimension five, so $(\mathcal{H}_3, g)$ has isometry group of the largest possible dimension for a non-constant curvature space.

In this paper we will fix a left invariant Riemannian metric in $\mathcal{H}_3$ and study the geometry of surfaces with special emphasis on minimal surfaces and the relationship with their Gauss map.

We have organized the paper as follows. Section 2 we present the basic geometry of the Heisenberg group, $\mathcal{H}_3$ including a basis for left invariant fields.

In Section 3 we study the non parametric surfaces in $\mathcal{H}_3$. We calculate the coefficients of the first and second fundamental form and the Gaussian curvature of this type of surfaces.

In Section 4 we present the Gauss map for hypersurfaces of any Lie group and present a relationship between this map and the second fundamental form and give a direct proof of a non existence of umbilical surfaces in $\mathcal{H}_3$.