JGSP **27** (2012) 115–122

JOURNAL OF
Geometry and Symmetry
in Physics

BOOK REVIEW

Pseudo-Riemannian Geometry, δ-Invariants and Applications, by Bang-Yen Chen, World Scientific, Singapore, 2011, xxxii + 477 pp., ISBN 978-981-4329-63-7.

The new book by Bang-Yen Chen, Professor from the Michigan State University is aimed to provide an extensive and comprehensive survey on pseudo-Riemannian submanifolds and the so-called δ-invariants, introduced by the author in the early 1990s, as well as on their applications. The book is both a good introduction to the theory of pseudo-Riemannian submanifolds and δ-invariants and a very useful reference to recent results in these areas, which have great value both within mathematics as well as in other natural sciences. Reading this book the readers will deepen their understanding and improve their appreciation of the concepts and theories under discussion.

The book consists of twenty chapters and can be divided into two parts. The first part, containing Chapter 1 through Chapter 12, provides an introduction to the subject of pseudo-Riemannian manifolds and their non-degenerate submanifolds, assuming from the reader some basic knowledge about manifold theory.

In Chapter 1 the author introduces the notion of a pseudo-Riemannian manifold and the basic notions regarding pseudo-Riemannian manifolds such as an affine connection, the Levi-Civita connection, parallel translation along a curve, geodesics, Riemannian curvature tensor, sectional, Ricci and scalar curvatures. Some basic propositions and theorems involving these notions are proved. The author reveals the physical interpretation of pseudo-Riemannian manifolds pointing out that spacetimes are the arenas in which all physical events take place. He explains the differential geometric viewpoint of the Kaluza-Klein theory and the use of higher dimensional pseudo-Riemannian manifolds, which lead to many new developments in string theory.

Chapter 2 is devoted to pseudo-Riemannian submanifolds. It starts with isometric immersions, the Cartan–Janet's and Nash's embedding theorems, continues with the Gauss' formula, the second fundamental form, the Weingarten's formula, the fundamental equations of Gauss, Codazzi and Ricci and the fundamental theorems of existence and uniqueness for pseudo-Riemannian submanifolds. In this chapter the reader can find the reduction theorem of Erbacher–Magid, two basic formulas for pseudo-Riemannian submanifolds (the formula of Beltrami and the formula of Chen), and some nice applications to minimal submanifolds and biharmonic submanifolds. The chapter ends with theorems providing a relationship between the squared mean curvature and the Ricci curvature of a pseudo-Riemannian submanifold, as well as a relationship between the shape operator and the Ricci curvature.

In Chapter 3 the author gives the main theorems regarding some special pseudo-Riemannian submanifolds. The simplest submanifolds are the totally geodesic submanifolds. The chapter starts with necessary and sufficient conditions for a pseudo-Riemannian submanifold to be totally geodesic. The totally geodesic submanifolds of a pseudo-Riemannian m-sphere and a pseudo-hyperbolic m-space are described. Nice references are given for the classification of totally geodesic submanifolds of symmetric spaces. Next, the classification of parallel pseudo-Riemannian submanifolds in indefinite real space forms is discussed. The classification of totally umbilical submanifolds and pseudo-umbilical submanifolds with parallel mean curvature vector in pseudo-Euclidean m-space, pseudo-Riemannian m-sphere and pseudo-hyperbolic m-space is also given. The author pays a special attention to minimal Lorentz surfaces and quasi-minimal Lorentz surfaces in indefinite space forms and gives classification results for some subclasses of such surfaces in the pseudo-Euclidean space $\mathbb{E}^4_2$, like quasi-minimal surfaces with parallel mean curvature vector, biharmonic quasi-minimal surfaces, quasi-minimal surfaces with constant Gauss curvature. A special section is devoted to marginally trapped surfaces and the important role they play in the theory of cosmic black holes.

One of the most fruitful generalizations of the direct product of two pseudo-Riemannian manifolds is the warped product, defined by Bishop and O'Neill. The notion of warped product plays an important role in differential geometry as well as in mathematical physics, especially in general relativity. Many basic solutions of the Einstein field equations are warped products. In the first part of Chapter 4 the author introduces the notion of a warped product of two pseudo-Riemannian manifolds and gives the expressions of the Levi-Civita connection, the curvature tensor and the Ricci tensor of a warped product manifold in terms of its warping function and the Levi-Civita connections, the curvature tensors and the Ricci tensors, respectively, of the base manifold and the fiber manifold. Twisted products provide another extension of direct products. The notion of twisted products generalizes the notion of warped products in a natural way, namely the warping function is replaced by twisting function, which depends on both factors. In the second part of the chapter the author defines the notions of a twisted product and a double-twisted product of two pseudo-Riemannian manifolds. Some basic propositions concerning twisted products and double-twisted products are given.

Chapter 5 is devoted to Robertson–Walker spacetimes. After explaining the relation between cosmology, Einstein's field equations and Robertson–Walker spacetimes, the author reveals the geometrical features of the Robertson–Walker spacetime. In general relativity, a the Robertson–Walker spacetime is a warped product of an open interval and a Riemannian three-manifold of constant curvature, where the warping function describes expanding or contracting of the Universe. In this chapter the author considers a Robertson–Walker spacetime as a warped product of an open interval and a real space form. Thus, using the properties of warped products and the results in the previous chapter, the author obtains basic properties of the Robertson–Walker spacetimes, describes the Robertson–Walker spacetimes of constant curvature, and classifies totally geodesic spacelike submanifolds, parallel submanifolds, totally umbilical submanifolds with parallel mean curvature vector, totally umbilical submanifolds with constant mean curvature, and hypersurfaces of constant curvature in Robertson–Walker spacetimes. At the end of the chapter the author provides explicit realizations of Robertson–Walker spacetimes as pseudo-Riemannian submanifolds in pseudo-Euclidean spaces.

In Chapter 6 the readers are provided with basic facts about the Hodge theory, the elliptic differential operators and the Jacobi's elliptic functions. The Hodge–de Rham decomposition theorem is given and some of its applications are presented. In a special section the author presents the spectra of some important Riemannian manifolds, namely, the unit n-sphere, the real projective n-space of constant curvature one, the complex projective n-space of constant holomorphic sectional curvature 4, the quaternion projective n-space of constant quaternionic sectional curvature 4, and the flat n-torus.

Chapter 7 concerns submanifolds of finite type. The notion of finite type submanifolds provides a natural way to combine the spectral geometry with the theory of submanifolds. The study of finite type submanifolds began in the late 1970s with the author's attempts to find the best possible estimates of the total mean curvature of a compact submanifold of Euclidean space. Since then many geometers have contributed to this theory. In this chapter the author introduces the notions of order and type of submanifolds of pseudo-Euclidean spaces. By applying the definitions of these notions he obtains some sharp estimates of the total mean curvature of compact submanifolds. Two useful criteria for determining whether a compact submanifold is of finite type are given – the minimal polynomial criterion and the variational minimal principle. A theorem giving the classification of 1-type submanifolds of pseudo-Euclidean m-space is presented. Next, the author determines all pseudo-Riemannian submanifolds of the pseudo-Euclidean m-space, the hyperbolic m-space and the de Sitter m-space, satisfying the condition that the mean curvature vector H is an eigenvector of the Laplacian Δ. Some results on biharmonic submanifolds, null 2-type submanifolds and spherical 2-type submanifolds are presented. The null 2-type spacelike surfaces with constant mean curvature and the null 2-type marginally trapped surfaces in Minkowski four-space are explicitly determined (up to a rigid motion). The chapter ends with the study of 2-type hypersurfaces in hyperbolic spaces. The reader is provided with a lot of references on the theory of submanifolds of finite type.

Chapter 8 deals with total mean curvature of submanifolds. The author gives some sharp relationships between the total mean curvature and the order of submanifolds of Euclidean space, as well as a sharp relationship between the total mean curvature and the first and second nonzero eigenvalues of the Laplacian for submanifolds of Euclidean space, the unit hypersphere, and the projective space.

Pseudo-Kähler manifolds are important in differential geometry as well as in mathematical physics, especially in string theory. Chapter 9 provides some basic properties of several special and interesting classes of pseudo-Riemannian submanifolds of a pseudo-Kähler manifold, namely complex submanifolds, purely real submanifolds, totally real submanifolds, CR-submanifolds, and slant submanifolds.

Para-Kähler geometry is the geometry related to the algebra of para-complex numbers, introduced as a generalization of complex numbers. Para-Kähler manifolds have been applied to supersymmetric field theory as well as to string theory in the recent years. In Chapter 10 the author defines the notions of a para-Kähler manifold and a para-Kähler space form and gives the properties of their curvature tensor and Ricci tensor. Next, he considers Lagrangian submanifolds of para-Kähler manifolds and formulates the fundamental existence and uniqueness theorems for Lagrangian submanifolds in para-Kähler space forms. The chapter also provides optimal results on the scalar curvature and the Ricci curvature of Lagrangian submanifolds of para-Kähler space forms. The last two sections deal with the properties of Lagrangian H-umbilical submanifolds and $\mathcal{P}R$-submanifolds of para-Kähler manifolds.

Chapter 11 is devoted to the pseudo-Riemannian submersions. The notion of submersions is a fundamental concept in differential topology. Riemannian submersions are submersions equipped with compatible Riemannian metrics and they are natural generalizations of warped products, which occur widely in geometry. Pseudo-Riemannian submersions were first introduced by O'Neill in his book *Semi-Riemannian Geometry with Applications to Relativity* (1983). They can be regarded as the duals of pseudo-Riemannian immersions. The chapter introduces the two fundamental tensors, characterizing the geometry of pseudo-Riemannian submersions and presents the properties of the O'Neill integrability tensor, as well as the so called O'Neill's equations for pseudo-Riemannian submersions. Pseudo-Riemannian submersions with totally geodesic fibers and pseudo-Riemannian submersions with minimal fibers are considered and many examples of such submersions are given. In the last section of this chapter the author presents the close relation between immersed submanifolds of the bundle space and the base manifold of a pseudo-Riemannian submersion.

Chapter 12 deals with contact metric manifolds and submanifolds. Contact geometry has broad applications in physics, especially in classical mechanics, dynamics, geometrical optics and control theory. Contact geometry studies a geometric structure on a manifold given by a hyperplane distribution in the tangent bundle and specified by a one-form satisfying a maximum nondegeneracy condition. A contact metric manifold is a contact manifold equipped with an associated metric, called a contact metric. An important class of contact metric manifolds is the class of the Sasakian manifolds. Sasakian manifolds have an associated vector field, called the characteristic vector field, which generates a one-dimensional foliation. Sasakian manifolds with Riemannian metric were introduced in 1960 by S. Sasaki and later the notion was extended by T. Takahashi to manifolds with pseudo-Riemannian metric. The chapter provides basic definitions concerning contact pseudo-Riemannian metric manifolds and Sasakian manifolds. The author presents the standard models of Sasakian space forms with definite metric and Sasakian space forms with indefinite metric. At the end of the chapter he discusses Legendre submanifolds and contact slant submanifolds via canonical fibration.

With Chapter 13 starts the second part of the book, which is devoted to the so-called δ-curvature invariants of a Riemannian manifold, introduced by the author in the early 1990s. These are new types of Riemannian invariants, different in nature from the classical Ricci and scalar curvatures. The main motivation of the author to introduce the δ-curvature invariants comes from one of the most fundamental problems in the theory of submanifolds – the immersibility (or non-immersibility) of a Riemannian manifold in an Euclidean space or, more generally, in a space form. In this chapter the author gives the definition of the δ-curvature invariants, which roughly speaking are obtained from the scalar curvature by deleting certain amount of sectional curvatures. Using these new invariants, the author characterizes Einstein spaces and conformally flat spaces. Next, he presents some fundamental inequalities involving δ-invariants, which give prima control on the most important extrinsic curvature, the squared mean curvature $\|H\|^2$, by the δ-invariants of Riemannian submanifolds in real space forms. Further, the author introduces the notion of an ideal immersion. The physical interpretation of an ideal immersion of a Riemannian manifold into a real space form is that the manifold feels the least possible amount of tension from the surrounding space at each point of the manifold. This is due to the inequality involving the squared mean curvature and the fact that the mean curvature vector field is exactly the tension field for isometric immersions. The author provides some examples of ideal immersions.

In Chapter 14 the author presents some applications of δ-invariants, based on the fundamental inequality giving optimal relation between δ-invariants and the squared mean curvature. One important application of δ-invariants is to provide solutions to the basic problem of minimal immersions: *given a Riemannian manifold M, what are the necessary conditions for M to admit a minimal isometric immersion in an Euclidean space?* The author gives applications of δ-invariants to spectral geometry, homogeneous spaces, warped products, Einstein manifolds, and conformally flat manifolds and provides a lot of examples. The last section of the chapter gives applications of δ-invariants to general relativity.

Chapter 15 provides applications of δ-invariants to Kähler and para-Kähler geometry. The author pays a special attention to Lagrangian immersions, proving some results concerning Lagrangian immersions into an Einstein Kähler manifold, a complex projective space of constant holomorphic sectional curvature, and a complex hyperbolic space. Another application of δ-invariants, given in this chapter, is to provide solutions to the following basic problem: *what are the necessary conditions for a compact Riemannian manifold to admit a Lagrangian isometric immersion into the complex Euclidean n-space?* In the next part of the chapter the author defines totally real δ-invariants of a Kähler manifold and gives their applications. He also provides a simple relationship between strongly minimal surfaces and framed-Einstein surfaces. Some non-trivial examples of strongly minimal Kähler surfaces are given. The chapter ends with applications of δ-invariants to real hypersurfaces of the complex hyperbolic m-space or complex projective m-space and applications to para-Kähler manifolds.

Chapter 16 is devoted to applications of δ-invariants to the contact geometry. A basic inequality involving δ-invariants for arbitrary submanifolds of Sasakian space forms is proved and an improved inequality for Legendre submanifolds of a Sasakian space forms is given. The author provides optimal inequalities for the scalar curvature and the Ricci curvature of Legendre submanifolds in Sasakian space forms. The notion of contact δ-invariants of an almost contact metric manifold is introduced and applications of contact δ-invariants is given.

In Chapter 17 the author provides applications of δ-invariants to affine geometry. After introducing the basic notions and formulas for affine hypersurfaces, centroaffine hypersurfaces, and graph hypersurfaces, he defines the so-called affine δ-invariants analogously to the definition of δ-invariants of Riemannian manifolds. In terms of these invariants an optimal inequality is proved for a definite centroaffine hypersurface in Euclidean space as well as for a graph hypersurface with positive definite Calabi metric. In order to show that these inequalities are the best possible, the author provides some examples. An interesting application is given in two propositions showing that there exist many warped products which can be realized either as graphs or as centroaffine hypersurfaces. In this way the author gives solutions to the following Realization Problem: *which Riemannian manifolds* (N, g) *can be immersed as affine hypersurfaces in an affine space such that the induced affine metric* h *on* N *is exactly the given Riemannian metric* g*?* For warped product centroaffine hypersurfaces and warped product graph hypersurfaces the author proves optimal inequalities and gives examples illustrating that these results are the best possible.

Chapter 18 presents applications of δ-invariants to Riemannian submersions. Introducing the notion of submersion δ-invariant, the author proves an optimal inequality for Riemannian submersions with totally geodesic fibers and provides some applications of this inequality. Classification results on Riemannian submersions which verify the equality case of the optimal inequality when the ambient space is a unit sphere are given. At the end of the chapter the reader can find two theorems providing links between Riemannian submersions and affine hypersurfaces via the submersion δ-invariant.

Chapter 19 provides some general results for real hypersurfaces and almost complex submanifolds of nearly Kähler manifolds. The best known example of a nearly Kähler manifold, but not Kählerian, is the unit six-sphere S_1^6 with the nearly Kähler structure induced from the vector cross product on the space of purely imaginary Cayley numbers. Some fundamental results for Lagrangian submanifolds of S_1^6 are presented and the classification of Lagrangian submanifolds of S_1^6 satisfying the Chen's basic $\delta(2)$-equality is given. Classification results for CR-submanifolds of the nearly Kähler S_1^6 satisfying the $\delta(2)$-equality are provided and Hopf hypersurfaces in S_1^6 are described. The last section of the chapter presents results concerning ideal real hypersurfaces of S_1^6.

In the last Chapter 20 of the book the author summarizes known results on $\delta(2)$-ideal submanifolds of space forms. Basic results on $\delta(2)$-ideal submanifolds of real space forms, reducible and minimal $\delta(2)$-ideal submanifolds of Euclidean space are presented. The classifications of $\delta(2)$-ideal tubes and $\delta(2)$-ideal isoparametric hypersurfaces in real space forms are given. The author also provides important theorems giving the classification of $\delta(2)$-ideal CMC hypersurfaces and conformally flat $\delta(2)$-ideal hypersurfaces in real space forms. Semi-symmetric, semi-parallel and pseudo-symmetric $\delta(2)$-ideal submanifolds of the Euclidean m-space are classified. The last part of the chapter provides results concerning $\delta(2)$-ideal

Lagrangian submanifolds, $\delta(2)$-ideal CR-submanifolds and $\delta(2)$-ideal Kähler hypersurfaces of complex space forms.

I strongly recommend the book *Pseudo-Riemannian Geometry, δ-Invariants and Applications* by Bang-Yen Chen to the mathematical community for its great value both in presenting the developments and the most important steps in the evolution of geometry of submanifolds in the last years, as well as for providing a very useful reference for all researchers interested in the theory of pseudo-Riemannian submanifolds and δ-invariants.

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