



BOOK REVIEW

Problems and Solutions for Groups, Lie Groups, Lie Algebras with Applications, by Willy-Hans Steeb, Igor Tanski and Yorick Hardy, World Scientific, Singapore 2012, x + 342 pp, ISBN 978-981-4383-90-5.

The authors come from the Johannesburg University, South Africa. One of them, Professor Willi-Hans Steeb is a managing director of the International School for Scientific Computing at the same University.

The purpose of the book, as announced in the Preface, is to supply a collection of problems in group theory, Lie group theory and Lie algebras. Each Chapter contains 100 completely solved problems.

Chapter 1 starts with the definitions of a group G . It also introduces the main notions such as subgroup, left and right cosets, the center $Z(G)$ of a group, as well as the centralizer of any subset $X \in G$.

After that the authors start to formulate problems, each one solved immediately. First they treat different types of group multiplications, some of which are rather unusual. Next, solving Problem 6 one finds that the complex numbers $q + ip\sqrt{5}$, where p and q are real rational numbers, form a group under multiplication of complex numbers. Similarly solving the next Problem 7 one realizes that the real numbers $r + s\sqrt{p}$, where r and s are rational numbers and p is a prime number, also form a group under the multiplication of real numbers.

The problems gradually become more and more involved which allows a beginner in the field to master the technicalities and to get a good knowledge in the field. Several of the problems concern the finite groups: the dihedral groups D_3 , D_4 , the cyclic groups C_3 , C_4 , the Kleinian group of order four. Thus one encounters their multiplication tables, characters and equivalence classes. The Chapter ends with two programming problems on Maxima with solutions and eight supplementary problems which are left to the reader.

The second Chapter is devoted to the Lie groups. It allows a diligent student to master and understand on simple examples a number of important facts about the Lie groups. The authors treat all important classical series of Lie groups $GL(n, \mathbb{C})$,

$SL(n, \mathbb{C})$, $O(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$ as well as their real forms $GL(n, \mathbb{R})$, $SU(n, \mathbb{C})$, $O(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$ etc., with $n = 2, 3$ and 4 . Special attention should be paid to the examples which discuss Cayley transform (Problem 3), Möbius transformation (Problem 8), the group commutator (Problems 4, 25 and 29), the characteristic equation, as well as finding the eigenvalues (Problem 37). Good attention is given to the orthogonal transformations (both complex and real) (see Problems 38, 48, 49, 53 and 91) and unitary transformations (Problems 43, 44, 57 and 85). A number of the problems allow one to understand the geometrical meaning of these transformations, mostly with the n -dimensional spheres \mathbb{S}^n , tori \mathbb{T}^n and hyperboloids (see Problems 47, 53, 63, 75, 76, 91, 92 and 94) and construction of their stereographic projections and atlases. Along the standard orthogonal transformations using trigonometric functions, the authors provide important examples of such transformations using Jacobi elliptic functions and Airy functions (Problems 69, 70 and 71). Additional important problems show how one can derive the Lie algebra generators and how to relate to each of them a vector field (Problems 10, 23, 29–35 and 95). Besides, it is demonstrated how to evaluate exponentials and other functions of matrices, the isomorphism between $SU(2)$ and its double covering of $SO(3)$, the notion of the tensor products, Hopf map and others.

The third Chapter is devoted to the Lie algebras. The authors start with the basic notions and definitions of Lie algebras such as subalgebras, ideals, adjoint representation, structure constants and their properties, Killing forms. They also define nilpotent, solvable, semisimple and simple Lie algebras. For the latter they briefly introduce the Cartan-Weyl basis, the notions of root, roots system, Cartan matrix and their basic properties.

Then they start with the problems. The first few of them aim at training abilities to tell a solvable algebra from a nilpotent or (semi)-simple ones. The Killing forms for all above-mentioned algebras are also calculated.

Most of the first 15 problems regard Lie algebras with three generators: the $\mathfrak{sl}(2)$ algebra and its real forms $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$. The authors include also proof of the simplicity of these algebras and derivation of its Killing form. Some of the following Problems (31 and 32) demonstrate that the Killing forms of non-semisimple Lie algebras are degenerated.

After that follow a series of problems treating the construction of Cartan-Weyl basis (Problems 15, 16, 37 and 38).

We mention also several other important topics:

- realization of Lie algebras in terms of vector fields (Problems 43 – 50, 70, 73, 74, 88, 90, 100 and 107 – 110). Special attention should be paid to Problem 70 where the algebra derived is the Virasoro algebra. Very instructive is

Problem 75 where the derived algebra is known as commutation relations of the spin-1 matrices that are realized by using vector fields, which represent the well known orbital moments in quantum mechanics;

- three-dimensional, four-dimensional and n -dimensional representation of $\mathfrak{sl}(2)$ (Problems 26, 27 and 28);
- tensor products of Lie algebras (Problems 51 – 57, 71, 81 and 84 – 88);
- isomorphisms between $\mathfrak{sl}(2)$, $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ (Problems 61 and 68).

Several problems treat the important issue in constructing the Casimir invariants. Here I would like to mention specifically Problem 97, in which the Casimir invariant is expressed in terms of vector fields.

A few of the problems at the end of this Chapter deal with some special issues of Lie algebra theory. Problem 94 demonstrates that using the quasi-multiplication $A \circ B = \frac{1}{2}(AB + BA)$ instead of the usual matrix multiplication allows one to obtain the so called Jordan algebras. Other problems treat Lie algebras over special fields of numbers, superalgebras (Problem 98) and the quantum $\mathfrak{sl}_q(2)$ algebra (Problem 99).

The last Chapter 4 is called Applications. It is really natural that it contains the most diverse set of problems, which corresponds to the long list of applications of the Lie groups in numerous fields of mathematics (dynamical systems, Hamiltonian systems, differential geometry) and physics (quantum mechanics, soliton theory, spin systems and others).

That is why I will mention here only a few of them. The authors treat several important types of finite groups, such as the icosahedron, Pauli and Clifford groups along with their q -bit extensions. In particular their multiplication tables and the conjugacy are calculated.

Several important types of dynamical systems are treated, including the Henon-Heiles and the Toda lattice, as well as anharmonic oscillators. Their Poisson brackets and Hamiltonian formulations are provided. In one occasion the authors also introduce the Moyal brackets. They also illustrate the notion of Liouville integrability.

Other problems involve deriving the Lax representations of several important integrable nonlinear evolution equations, such as the principal chiral field equations, the Landau-Lifschitz equations and others.

Several of the problems deal with constructing the algebra of symmetries of a given (partial) differential operator. At the end I should mention the braid-like relations and Knizhnik-Zamolodchikov equation.

Finally, the task to make the students familiar with the Lie algebras using only four Chapters with 100 problems in each is indeed too ambitious. Nevertheless the authors have made a very good selection of problems in each of the Chapters, illustrating the most important results from the theory of Lie groups and Lie algebras. An additional merit is that each of the Chapters is extended by a few programming problems in C++, which helps in treating similar problems for higher rank algebras and groups.

In all cases the problems are well posed and instructive. Their sequence is pedagogical. I highly recommend this book as an excellent introduction to the Lie group and Lie algebra theory and related subjects such as (integrable) dynamical and Hamiltonian systems, differential geometry, symmetries of differential operators, soliton theory and etc. More complete treatment of these and other parts of the beautiful theory of Lie algebras and Lie groups can be found in the books listed below.

References

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Vladimir Gerdjikov
Institute of Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
Blvd. Tsarigradsko chaussée 72, Sofia 1784
BULGARIA
E-mail address: gerjikov@inrne.bas.bg