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BOOK REVIEW

Regularity Theory for Mean Curvature Flow, by Klaus Ecker, Progress in Nonlinear Differential Equations and Their Applications, vol. 57, Birkhäuser, Boston, xi + 165 pp, ISBN 0-8176-3243-3

As the author had stated in the *Preface*, this book was started as a set of informal notes on Brakke's theory [1] about mean curvature flow. The aforementioned theory is concerned with the motion of surfaces for which the normal velocity at every point is given by the mean curvature vector at that point. From some more abstract point of view this is an analogue of heat flow generated by a purely geometric process.

Actually, under the mean curvature flow, surfaces usually develop singularities in a finite time period and this book presents the necessary techniques for the study of singularities of the mean curvature flow. It is largely based on the work done by Brakke, but some more recent developments (mainly due to the author) are also incorporated. One can say that the book is focussed on the special case where smooth solutions of mean curvature flow develop singularities for the first time, thus expressing underlying ideas almost entirely in the language of differential geometry and partial differential equations. Notation from geometric measure theory (see [2] and [4] for details) were kept here to a minimum.

Just to help the reader to orientate himself in the topics discussed in this work we present below the Table of Contents of the book, namely:
Preface – Introduction – Special Solutions and Global Behaviour – Local Estimates via the Maximum Principle – Integral Estimates and Monotonicity Formulas – Regularity Theory at the First Singular Time – Geometry of Hypersurfaces – Derivation of the Evolution Equations – Background on Geometric Measure Theory – Local Results for Minimal Hypersurfaces – Remarks on Brakke's Clearing Out Lemma – Local Monotonicity in Closed Form – Bibliography – Index.

After presenting the preliminary material and main concept of mean curvature flow, this self-contained and systematic exposition aims to cover in some depth the essential details of the techniques leading to a proof of Brakke's main regularity theorem. The latter one is illustrated with important examples and special solutions including a detailed discussion of homothetic solutions. Then, the local point-wise estimates on geometric quantities for smooth solutions of mean curvature flow are derived in a streamlined presentation. The exposition continues with rescaling methods, monotonicity formulas, mean value inequalities and ends with two local regularity theorems and establishing an estimate about the singular set. A chapter by chapter description of the book follows below.

Chapter 2 introduces the concept of mean curvature flow and illustrates several important examples and special solutions. Homothetic solutions, which play a central role in the regularity theory, are covered in significant detail. Several global results are also stated, which describe long-term existence and asymptotic behaviour in special situations. In the cases of convex initial data and of embedded curves, no singularities form before the solution disappears. Entire graph solutions never develop singularities.

Local point-wise estimations on geometric quantities for smooth hypersurfaces moving by mean curvatures are derived in *Chapter 3*. First, control on the position vector of the moving surfaces is established. This leads in particular to conditions on an initial hypersurface that guarantee the formation of singularities before the solution disappears.

The behaviour of the integral quantities is studied in *Chapter 4* starting with an integral version of mean curvature flow, which also serves as the basis for Brakke's weak solution concept. The main result is Huisken's monotonicity formula [3], for which a new localised version is derived. Consequences of these include upper and lower bounds on the area ratio inside balls, as well as local mean value inequalities. Further, an explanation is given on how homothetically shrinking solutions arise as limits of rescaled solutions. Although all results are formulated in the smooth case, most of the proofs given there are easily adapted to Brakke solutions.

Chapter 5 contains the actual regularity theory. After introducing a number of concepts from geometric measure theory, Brakke's main regularity theorem is stated and the central hypothesis of his result is discussed. The technical part of this chapter begins with two local regularity results. The first version is due to White [5] and uses the monotonicity formula as the essential tool. The second version is a new result and employs a form of L^2-height deviation from a hyperplane and is closer in spirit to Brakke's original one.

For the convenience of the reader an *Appendix* is added. Its first three parts (i.e., *Appendices A-C*) list in more detail the definitions and facts for hypersurfaces in

Euclidean space used throughout the text, give a derivation of the basic evolution equations for mean curvature flow and provide some background on the geometric measure theory used in the book. The remaining three parts (*Appendices D-F*) present material which is related to but not essential for the main part of the book. *Appendix D* provides an account of various fundamental techniques in minimal surface theory such as monotonicity and mean value formulas for smooth hypersurfaces as well as minimal surface versions of the regularity proofs in *Chapter 5*. *Appendix E* gives a proof of a stronger version of Brakke's clearing out lemma than the one used in the proof of Brakke's main regularity theorem given in *Chapter 5*. *Appendix F* gives the derivation of a new local monotonicity formula due to the author which is analogous to the monotonicity formula for minimal hypersurfaces. In particular, in this formula, space and time are combined in a geometrically natural fashion.

As one can recognize from all above, the target readers are graduate students and researchers in nonlinear PDEs, differential geometry, geometric measure theory and mathematical physics in general. The bibliography counts over seventy sources followed by a carefully prepared index and probably the book will serve as a value reference on the subject for many years.

References

[1] Brakke K., *The Motion of a Surface by its Mean Curvature*, Princeton Math. Notes, Princeton University Press, Princeton, New Jersey, 1978.

[2] Federer H., *Geometric Measure Theory*, Springer, Berlin, 1969.

[3] Huisken G., *Asymptotic Behaviour for Singularities of the Mean Curvature Flow*, J. Diff. Geom. **31** (1990) 285-299.

[4] Morgan F., *Geometric Measure Theory: A Beginners Guide*, Academic Press, Boston, 1988.

[5] White B., *A Local Regularity Theorem for Classical Mean Curvature Flow*, Preprint (2000).

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