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BOOK REVIEW

Electricity and Magnetism for Mathematicians: *A Guided Path From Maxwell's Equations to Yang-Mills*, by Thomas A. Garrity, Cambridge University Press, New York 2015, xiv+282 pp., ISBN 978-1-107-07820-8 Hardback, ISBN 978-1-107-43516-2 Paperback.

A modern mathematical physicist could hardly imagine doing serious studies in the frame of contemporary theoretical and mathematical physics without making substantial use of concepts like vector and tensor fields, differential forms, Stokes theorem on manifolds, Lie derivative and symmetries, Lie (super)groups and Lie (super)algebras and representations, bundles with connection, curvature, Hamiltonian and Poisson structures, Frobenius integrability, and other directions of modern differential geometry. Any physical system has spatial structure and shows definite stability properties, so, it can support its existence and compensate in definite degree the external disturbances through appropriate shape changes and kinematical behavior without losing identity. Shortly speaking, its time existence is a dynamical process being strongly connected with various and continuous internal and external stress-energy-momentum exchange processes. All these processes are real phenomena and any attempt for their description should be based on appropriate mathematical structures. The more than a century intensive interaction between differential geometry and the theoretical and mathematical physics turned out to be exclusively useful, suggestive and creative process. The author of the book under review, clearly underlines that mathematics is not just abstract thinking, and physics is not just working in labs, and the page 6 of his book demonstrates this by the motto

$$Physics \rightarrow Mathematics.$$

It could be said that physics has always stimulated mathematics to develop new branches, and mathematics has always been very responsive and warm-hearted to the needs of physics.

The book under review consists of 21 Chapters, Bibliography and Index.

The first Chapter, pp.1-8, gives a very short history of the subject, including some moments of the 20th century quantum developments. Chapters two and three present the original Maxwell's equations and electromagnetic waves. Chapters 4,5,7 introduce the relativistic view, Chapters six and eight give the Lagrangian approach. The real relativistic form of Maxwell's equations, making use of differential forms and Hodge *-operator is given in Chapters 9-11. Then in Chapters 12-15 the author presents the necessary quantum formalism for quantifying Maxwell's equations. In the following four chapters modern differential geometry (manifolds, vector bundles, connections and curvature) is appropriately introduced. Finally, the last two chapters make use of this language to represent the subject in modern terms. It is very important to specially note the great number of problems, most of which are explicitly solved.

Let us now have some glances to the contents of some chapters.

In Section 2.1 it is said that "Maxwell's equations link together three vector fields and a real-valued function", then the components of the vector fields are given as functions of the space coordinates (x, y, z) and the time t in a way that these independent variables are of the same significance, which is not quite so: time t is *external* to the space coordinates parameter, e.g., the differential of the charge density ρ, when considered as differential one-form on $\mathbb{R}^3$, will not contain the derivative with respect to t. So, in my view, the notation $\rho(x, y, z, t)$ should be written and understood somehow like $\rho(x, y, z; t)$. This is why the time derivative of the two fields does not change their vector field nature. This notion of time should be respected, in fact, in all pre-relativistic mathematical physics.

Another remark concerns the different nature of the electric and magnetic fields and the nature of charge density: introducing mathematical fields as formal images of physical objects, should depend on the regions of their definition, so, the correct definition of mathematical images of continuous physical objects should depend on the *physical* nature of the region: vacuum, or continuous distribution of charges. In theoretical physics the two definitions are not the same.

It also deserves to note the notion for "force", as used in the book. In mechanics, roughly speaking, force means "*one mass body directly acts on another mass body*", i.e., there is no distance between them. In electrodynamics the mutual action between the two charged bodies is realized through fields, so, the electric fields of the two charged bodies interact somehow, and this interaction changes each of the two fields, so, in order to recoup its losses, each body changes its state of motion. The same is the situation in classical gravitation theory, e.g., the Earth and the Moon could hardly feel each other if the two gravitational fields were absent, or, if they do not interact, which is hardly believable in view of the same physical structure they have. Physical fields are physical objects, so, they carry stress-energy-momentum, and if two such fields interact, then if the integral interaction energy is negative, *attraction* between the charges/masses is observed, the case of positive interaction energy means corresponding repulsion observed. Electrodynamical field objects, however, show a different way of keeping time stability: the corresponding stress-energy-momentum tensor does NOT contain nonzero interaction energy between the electric and magnetic constituent fields, nevertheless, energy exchange takes place, but each field simultaneously loses and gains the *same* quantities of energy, and this special kind of interaction is responsible for the quite specific nature of photon-like objects.

In this direction it seems also appropriate to have in mind that *static* fields can not be considered as *acting* force fields, except when they are equal in magnitude, but act in opposite direction and establish dynamical equilibrium at the point under consideration. In the case of reducing the force field on a trajectory, then usually the coordinate on the trajectory is identified with time t, so the force field is no more static. This should be in mind when variational approach in mechanics is under consideration.

It seems also important to have clear understanding of the role of zero divergences of the electric and magnetic fields in the vacuum case. The sense of a zero divergence of a vector field X on $\mathbb{R}^3$ is that the Lie derivative $L_X\omega = \mathrm{d}i_X\omega + i_X\mathrm{d}\omega$ of the volume form $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ along X is zero, and since $\mathrm{d}\omega = 0$, the two-form $i_X\omega$ is closed: $\mathrm{d}i_X\omega = \mathrm{div}(X).\omega = 0$. This fact generates one-form (α, β)-potentials in the vacuum case: $i_{\mathbf{E}}\omega = \mathrm{d}\alpha$ and $i_{\mathbf{B}}\omega = \mathrm{d}\beta$. Now, in order to model mutual and *time-realizable* interaction between $\mathbf{E}$ and $\mathbf{B}$ making help of α and β, Maxwell's equations require

$$\frac{1}{c}\frac{\partial}{\partial t} i_{\mathbf{E}}\omega = \mathrm{d}\beta, \qquad \frac{1}{c}\frac{\partial}{\partial t} i_{\mathbf{B}}\omega = -\mathrm{d}\alpha.$$

These two vector equations formally introduce a kind of *local interaction* between the *divergence-free nature of* $(\mathbf{E}, \mathbf{B})$ and the *external* nature of their possible time dependence.

Three more notes.

First, young physically inclined mathematicians must not forget that plane electromagnetic waves are *exact* solutions of Maxwell's charge free equations, and each such solution carries *infinite* integral energy, so, there is no way to physically create such waves. This follows from the fact that every component of the two fields must satisfy D'Alembert wave equation.

Second, the charge free Maxwell equations generate eight scalar equations for six functions, i.e., the corresponding system is overdetermined.

Third, it could be useful for the readers to have information about the existent nonlinearizations of Maxwell's equations.

In my view, the whole mathematical part of the book is well and appropriately presented, including the bundle theory part, the notions are very well explained and very well illustrated by the many well chosen and well ordered examples.

My whole impression is that all basic material, that is necessary for a young mathematical student in the field of geometrical formulation of mechanics, electrodynamics and first steps in quantum theory, is ordered and represented in an appropriate manner with a corresponding respect to the reader. The problems, exercises and hints help the reader to test himself in the right understanding of what he has already read. So, the book may be recommended to all young students being interested in acquiring first steps to finding correspondence between harmony in the physical world and harmony in corresponding mathematical structures.

Stoil Donev
Institute for Nuclear Research
and Nuclear Energy
Bulgarian Academy of Sciences
Blvd. Tzarigradsko chaussee 72
1784 Sofia, BULGARIA
E-mail address: sdonev@inrne.bas.bg