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# BOOK REVIEW

*A Spinorial Approach to Riemannian and Conformal Geometry*, by Jean-Pierre Bourguignon, Oussama Hijazi, Jean-Louis Milhorat, Andrei Moroianu and Sergiu Moroianu, European Mathematical Society, Switzerland 2015, ix+452 pp, ISBN 978-3-03719-136-1

The Dirac equation (in natural units $c = \hbar = 1$)

$$\mathrm{i}\sum_{\nu=0}^{3}\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0$$

was formulated by P.A.M. Dirac in 1928. It is one of the most fundamental equations of quantum mechanics. To be explicit in this equation $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\psi(x) = \psi(x^0, \ldots, x^3) \in \mathbb{C}^4, x \in \mathbb{R}^{1,3}$, is a complex four-vector valued function on $\mathbb{R}^{1,3}[x^0, x^1, x^2, x^3]$, called a spinor field, and $\gamma^\mu$ are $4 \times 4$ matrices, the so-called Dirac's $\gamma$-matrices. They satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu}$ is the Minkowski tensor. The Dirac equation is invariant with respect to translations and a specific action of the connected double cover of the Lorentz group $\mathrm{O}(1,3)$ called the **spinor representation**. Applying the operator $\mathrm{i}\sum_{i=0}^{3}\gamma^\mu\partial_\mu + m$ to the Dirac equation, one obtains the so-called Klein–Gordon equation $\Box\psi + m^2\psi = 0$, where $\Box = \partial^2_{x^0} - \partial^2_{x^1} - \partial^2_{x^2} - \partial^2_{x^3}$ is the wave operator. Let us note that a kind of spinors was already known to Pauli who postulated his equation similar to the Dirac equation in the realm of non-relativistic quantum mechanics. Although its relevance is rather limited in quantum theory, the Dirac equation is still a helpful tool in physics and a source of inspiration for analysts and geometers. In mathematics, one usually considers Riemannian manifolds rather than the Lorentzian ones. The appropriate Dirac operator is adapted to the Riemannian metric. Its square is an elliptic operator of Laplace type. In the reviewed book, the authors consider Riemannian manifolds and appropriate elliptic Dirac operators. Spinors and Dirac operators are also used in non-commutative geometry of A. Connes, serve as an important example for applications of the Atiyah–Singer index theorem, and a spectrum of the Dirac operator is helpful in determining of topological properties of the underlying manifold.

**Spinors** are elements of spinor representation spaces of the connected double cover $\mathrm{Spin}_n$ of the special orthogonal group $\mathrm{SO}_n$, $n \geq 2$. These double covers are called the **spin groups**. The spinor representations are realized in the **Clifford algebra**. For a vector space $V$ equipped with a symmetric bilinear form $B$, one defines the Clifford algebra as the tensor algebra of $V$ divided by the ideal generated by $v \otimes w + w \otimes v + B(v, w)1$, i.e.,

$$\mathrm{Cliff}(V, B) = T(V)/\langle v \otimes w + w \otimes v + B(v, w)1 | \, v, w \in V \rangle.$$

The spin group is generated by the length-one elements in $\mathrm{Cliff}(V, B)$ where the bilinear form of $B$ is extended to $\mathrm{Cliff}(V, B)$ multiplicatively. Spinor representations are specific one-sided ideals in the Clifford algebra. The spinor group acts on these ideals by restriction. This action is the spinor representation. **Spinor fields**, sometimes called just spinors, are smooth sections of spinor bundles. The spinor bundles are constructed via the so-called **spin structures** which are specific principal $\mathrm{Spin}_n$-bundles over the underlying Riemanian manifold. Spinor bundles are vector bundles associated to these principal bundles via the spinor representation. One often considers several 'derivatives' of the principal bundles, e.g., complex or conformal spin structures. The Dirac operator is a certain first order differential operator on the spinor fields. It is constructed with help of the Levi-Civita connection of the Riemannian manifold and the so-called spinor multiplication. The existence of a spin structure is a purely topological matter. It is formulated by characteristic classes.

An important tool in spin geometry is the so-called Schrödinger–Lichnerowicz formula. This formula relates the square of the Dirac operator to the Laplace operator on spinors constructed out of the Levi-Civita connection. It is parallel to the well known Weitzenböck formula for the difference of the deRham Laplace operator and the Levi-Civita Laplace operator on functions or, more generally, on exterior differential forms.

A traditional topic in the theory of Dirac operators are the lower bound estimates for eigenvalues of the Dirac operator and especially, for the smallest one. The Lichnerowicz's, Friedrich's, Hijazi's, Kirchberg's (Kähler case), Kramer's, Semmelmann's and Weingart's (quaternion-Kähler case) lower bound estimates for the first eigenvalue of the Dirac operator on compact spin manifolds are examples of some of them. The limiting cases of the estimates, i.e., eigenfunctions for which the minimal eigenvalues are achieved, satisfy equations connected to the so-called Penrose twistor operator and to the Killing spinor equation. The Lorentzian counterparts of these equations can often be found in physics related to string theory or to super-symmetry. On the other hand, it is often known for which manifolds these limiting cases are achieved. This may give answers to physicists on which manifolds the Riemannian counterparts of their equations are satisfied.

A further theme often considered are the spectra of the Dirac operators on symmetric spaces and spectral computations for the so-called model spaces (spheres, complex and quaternion projective spaces). Spectra of the Dirac operators on symmetric spaces and on the model spaces may be computed by an algebraic manner using the representation theory of appropriate Lie groups. The tools used include the Peter–Weyl and the highest weight theorem.

The book under review is structured in four parts and it is divided into 15 chapters. Part I starts with algebraic aspects of spinors, namely with the notions of Clifford algebras and their classification, orthogonal and spin groups, and spinor representations as well as with fundamental representations of the special orthogonal groups. Second chapter of Part I is devoted to spin structures – a base for introducing spinor fields on manifolds. To be more specific, spinorial, spin, spin$^c$ manifolds, spinor bundles and spin connections are introduced. It contains a treatment on conformal and Weyl structures as well. This chapter contains also a treatment on first order invariant operators on spinor bundles, sometimes called Stein–Weiss operators or gradients. Especially, the Dirac and the Penrose twistor operators are introduced there. The Schrödinger–Lichnerowicz formula is derived in this chapter as well. In the third chapter, Čech cohomology and the Stiefel–Whitney classes are used for expressing a sufficient and necessary condition for the existence of a spin structure. Fourth chapter is devoted to analysis on manifolds (Fourier transform on manifolds, pseudodifferential operators, Sobolev spaces, elliptic operators, parametrix and the Atiyah–Singer index theorem).

Part II contains eigenvalue estimates for the Dirac operator. It consists of formulations and proofs of the inequalities mentioned above as well as of several of their consequences. Let us note that in the proof of the Friedrich's inequality, the special covariant derivative considered originally is not used. Part II is divided into three chapters devoted to the cases of Riemannian, Kähler and quaternion-Kähler manifolds. Conformal covariance of the Dirac operator is used several times. For a related broad use of the conformal symmetry see [4]. This part contains also a proof of the Witten's positive mass theorem and a systematic treatment of identities of Weitzenböck's type.

Part III is devoted to the so-called special spinor fields. In particular, parallel spinors on spin and spin$^c$ manifolds and manifolds admitting Killing spinors are treated. The existence of these specific spinors implies a reduction of the holonomy group. Cone constructions are helpful to formulate conditions which are satisfied by these special fields. The existence of real Killing spinors leads to manifolds with an additional structure such as the Sasaki–Einstein, nearly Kähler, weak $G_2$, and several other types. The special spinor fields are investigated for conformal, Kähler and quaternion-Kähler manifolds in individual chapters. There is a part devoted to the Cauchy problem for Einstein metrics and to generalized Killing spinors.

More than hundred of pages are devoted to the spectra of the Dirac operator on symmetric spaces and especially, on the model ones (Part IV). It starts with a comprehensive introduction to the representation theory of compact Lie groups. Cartan algebras and roots of simple Lie algebras as well as maximal tori and characters of simple Lie groups are introduced. The representation theory is explained carefully and, so to speak, from the beginning. It covers the Peter–Weyl theorem, Frobenius reciprocity, Cartan's weight theory and also the theorem on the highest weight. Developed facts are used to derive parameterizations of all finite dimensional irreducible representations of the special unitary groups $\mathrm{SU}_n$, unitary groups $\mathrm{U}_n$, spin groups $\mathrm{Spin}_n$ and the group $\mathrm{Sp}_n$ of unitary symplectic $2n \times 2n$ matrices (the quaternionic symplectic group). Basic facts from harmonic analysis are explained by the authors and used for the spectra computing. Model spaces are represented as symmetric spaces with appropriate Riemannian structures. At the end, the well known spectra of the Dirac operator on spheres, complex and quaternion projective spaces are computed explicitly and some results on further spaces are recalled.

The book consists of more than 450 pages. In the last four decades, several monographs devoted to Dirac operators on manifolds were published. Let us mention the books of Baum [1], Lawson and Michelsohn [5], Friedrich [2], and Ginoux [3]. Besides different approaches to proofs of several theorems and the use of the conformal covariance, the width of topics and the self-contained approach make the reviewed monograph distinguished. Moreover, the treated topics can be read independently to a large extent. The authors present a comprehensive and up-to-date view on the Riemannian spinors and the classical Dirac operators. A further development in the direction of Dirac operators for connections with torsion and Lorentz Dirac's operators seem not to be touched. On the other hand, a one single book can hardly cover all topics connected to the main theme.

Since basic results and methods from the auxiliary areas, such as of global analysis, algebraic topology, and representation theory, are explained carefully, the book may be used by a wide spectrum of students and researchers in physics and mathematics. Only a rather basic knowledge on principal bundles and connections is required. Those who would like to start a research in this area may benefit from the treatment embracing a broad variety of topics of the spinor theory and going into detail as well. Researchers working in non-commutative geometry or global analysis can use the book as a source of examples illuminating the 'abstract' statements typical for these research areas. We hope that the reader will enjoy the book – its clear proofs, broad width of topics, and last but not least the self-contained and reader-friendly approach of the authors, who are prominent experts in the field of Riemannian geometry and spinors.

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