JGSP **50** (2018) 1–10

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

LORENTZ-INVARIANT SECOND-ORDER TENSORS AND AN IRREDUCIBLE SET OF MATRICES

MAYEUL ARMINJON

Communicated by Ivaïlo M. Mladenov

Abstract. We prove that, up to multiplication by a scalar, the Minkowski metric tensor is the only second-order tensor that is Lorentz-invariant. To prove this, we show that a specific set of three 4×4 matrices, made of two rotation matrices plus a Lorentz boost, is irreducible.

MSC: 15A18, 83A05
Keywords: Irreducible set of matrices, linear algebra, Lorentz group, Minkowski metric

1. Introduction

It is a basic result of special relativity that the Minkowski metric tensor is invariant under the Lorentz group. The main aim of this paper is to prove that, up to a scalar, this property characterizes the Minkowski metric

Theorem 1. *Let* $(\mathrm{M}, \boldsymbol{\gamma}^0)$ *be the four-dimensional Minkowski spacetime. Any* $(0, 2)$ *tensor on* M *that is invariant under the Lorentz group is a scalar multiple of the Minkowski metric tensor* $\boldsymbol{\gamma}^0$.

(See Note 1 for the extension to a Lorentzian spacetime.) This result is not very surprising and seems to be heuristically known. For instance, after having introduced the classical totally antisymmetric fourth-order tensor, Maggiore [3, p. 24] states: "The only other invariant tensor of the Lorentz group is $\eta_{\mu\nu}$ as its invariance follows from the defining property of the Lorentz group, equation (2.13)." (The latter equation is equivalent to equation (3) below.) Nevertheless, we saw neither a precise statement of the above Theorem nor a correct proof in the literature that we could find. The proof that we present here appeals to Schur's lemma (Section 2). However, to identify a relevant irreducible set of 4×4 matrices in order to use Schur's lemma was not completely obvious. To prove the irreducibility of that set S, we had to study in detail which are the invariant subspaces of each of the matrices that constitute S (Section 3). Although it is often easy to check that some

doi: 10.7546/jgsp-50-2018-1-10