Supplementary Material

Soliton Hierarchies Associated with Lie Algebra $\mathfrak{sp}(6)$

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Supplement 1. The Remaining Structure Constants of $\mathfrak{sp}(6)$

A basis for the Lie algebra $\mathfrak{sp}(6)$ is given by 1) and its structure constants are as follows

$$
\begin{aligned}
&[E_1, E_4] = E_4,\ [E_1, E_5] = -E_5,\ [E_1, E_6] = E_6,\ [E_1, E_7] = -E_7\\
&[E_1, E_{10}] = E_{10},\ [E_1, E_{11}] = -E_{11},\ [E_1, E_{14}] = 2E_{14},\ [E_1, E_{15}] = -2E_{15}\\
&[E_1, E_{16}] = E_{16},\ [E_1, E_{17}] = -E_{17},\ [E_2, E_4] = -E_4,\ [E_2, E_5] = E_5\\
&[E_2, E_8] = E_8,\ [E_2, E_9] = -E_9,\ [E_2, E_{12}] = 2E_{12},\ [E_2, E_{13}] = -2E_{13}\\
&[E_2, E_{16}] = E_{16},\ [E_2, E_{17}] = -E_{17},\ [E_2, E_{18}] = E_{18},\ [E_2, E_{19}] = -E_{19}\\
&[E_3, E_6] = -E_6,\ [E_3, E_7] = E_7,\ [E_3, E_8] = -E_8,\ [E_3, E_9] = E_9\\
&[E_3, E_{10}] = E_{10},\ [E_3, E_{11}] = -E_{11},\ [E_3, E_{18}] = E_{18},\ [E_3, E_{19}] = -E_{19}\\
&[E_3, E_{20}] = 2E_{20},\ [E_3, E_{21}] = -2E_{21},\ [E_4, E_5] = E_1 - E_2,\ [E_4, E_7] = -E_9\\
&[E_4, E_8] = E_6,\ [E_4, E_{11}] = -E_{19},\ [E_4, E_{12}] = E_{16},\ [E_4, E_{15}] = -E_{17}\\
&[E_4, E_{16}] = 2E_{14},\ [E_4, E_{17}] = -2E_{13},\ [E_4, E_{18}] = E_{10},\ [E_5, E_6] = E_8\\
&[E_5, E_9] = -E_7,\ [E_5, E_{10}] = E_{18},\ [E_5, E_{13}] = -E_{17},\ [E_5, E_{14}] = E_{16}\\
&[E_5, E_{16}] = 2E_{12},\ [E_5, E_{17}] = -2E_{15},\ [E_5, E_{19}] = -E_{11}\\
&[E_6, E_7] = E_1 - E_3,\ [E_6, E_9] = E_4,\ [E_6, E_{10}] = 2E_{14},\ [E_6, E_{11}] = -2E_{21} \qquad \text{(S1)}\\
&[E_6, E_{15}] = -E_{11},\ [E_6, E_{17}] = -E_{19},\ [E_6, E_{18}] = E_{16},\ [E_6, E_{20}] = E_{10}\\
&[E_7, E_8] = -E_5,\ [E_7, E_{10}] = 2E_{20},\ [E_7, E_{11}] = -2E_{15},\ [E_7, E_{14}] = E_{10}\\
&[E_7, E_{16}] = E_{18},\ [E_7, E_{19}] = -E_{17},\ [E_7, E_{21}] = -E_{11},\ [E_8, E_9] = E_2 - E_3\\
&[E_8, E_{10}] = E_{16},\ [E_8, E_{13}] = -E_{19},\ [E_8, E_{17}] = -E_{11},\ [E_8, E_{18}] = 2E_{12}\\
&[E_8, E_{19}] = -2E_{21},\ [E_8, E_{20}] = E_{18},\ [E_9, E_{11}] = -E_{17},\ [E_9, E_{12}] = E_{18}\\
&[E_9, E_{16}] = E_{10},\ [E_9, E_{18}] = 2E_{20},\ [E_9, E_{19}] = -2E_{13},\ [E_9, E_{21}] = -E_{19}\\
&[E_{10}, E_{11}] = E_1 + E_3,\ [E_{10}, E_{15}] = E_7,\ [E_{10}, E_{17}] = E_9,\ [E_{10}, E_{19}] = E_4\\
&[E_{10}, E_{21}] = E_6,\ [E_{11}, E_{14}] = -E_6,\ [E_{11}, E_{16}] = -E_8,\ [E_{11}, E_{18}] = -E_5\\
&[E_{11}, E_{20}] = -E_7,\ [E_{12}, E_{13}] = E_2,\ [E_{12}, E_{17}] = E_5,\ [E_{12}, E_{19}] = E_8\\
&[E_{13}, E_{16}] = -E_4,\ [E_{13}, E_{18}] = -E_9,\ [E_{14}, E_{15}] = E_1,\ [E_{14}, E_{17}] = E_4
\end{aligned}
$$

$$[E_{15}, E_{16}] = -E_5,\ [E_{16}, E_{17}] = E_1 + E_2,\ [E_{16}, E_{19}] = E_6,\ [E_{17}, E_{18}] = -E_7$$
$$[E_{18}, E_{19}] = E_2 + E_3, [E_{18}, E_{21}] = E_8,\ [E_{19}, E_{20}] = -E_9,\ [E_{20}, E_{21}] = E_3.$$

Supplement 2. The Recurrence Relation of $V_{0,x} = [U_0, V_0]$

In the isospectral problem studied in Section 2.1, by choosing the spectral matrices (2) and (3), the stationary zero-curvature equation $V_{0,x} = [U_0, V_0]$ yields the following recurrence relation

$$\begin{cases} a_x = u_1 l - u_2 k + u_5 q - u_6 p + u_7 s - u_8 r \\ b_x = u_3 o - u_4 m + u_7 s - u_8 r + u_9 u - u_{10} t \\ c_x = u_1 l - u_2 k + u_9 u - u_{10} t + u_{11} w - u_{12} v \\ d_x = u_1 u - u_4 r + u_5 s + u_7 o - u_8 p - u_{10} k \\ e_x = -u_2 t + u_3 s - u_6 r + u_7 q - u_8 m + u_9 l \\ f_x = u_1 w - u_2 p + u_5 l + u_7 u - u_{10} r - u_{12} k \\ g_x = u_1 q - u_2 v - u_6 k - u_8 t + u_9 s + u_{11} l \\ h_x = -u_2 r + u_3 u + u_7 l + u_9 w - u_{10} m - u_{12} t \\ j_x = u_1 s - u_4 t - u_8 k + u_9 o - u_{10} v + u_{11} u \\ k_x = 2\lambda k - u_1 a - u_1 c - u_5 g - u_7 j - u_9 d - u_{11} f \\ l_x = -2\lambda l + u_2 a + u_2 c + u_6 f + u_8 h + u_{10} e + u_{12} g \\ m_x = 2\lambda m - 2u_3 b - 2u_7 e - 2u_9 h \\ o_x = -2\lambda o + 2u_4 b + 2u_8 d + 2u_{10} j \\ p_x = 2\lambda p - 2u_1 f - 2u_5 a - 2u_7 d \\ q_x = -2\lambda q + 2u_2 g + 2u_6 a + 2u_8 e \\ r_x = 2\lambda r - u_1 h - u_3 d - u_5 e - u_7 a - u_7 b - u_9 f \\ s_x = -2\lambda s + u_2 j + u_4 e + u_6 d + u_8 a + u_8 b + u_{10} g \\ t_x = 2\lambda t - u_1 e - u_3 j - u_7 g - u_9 b - u_9 c - u_{11} h \\ u_x = -2\lambda u + u_2 d + u_4 h + u_8 f + u_{10} b + u_{10} c + u_{12} j \\ v_x = 2\lambda v - 2u_1 g - 2u_9 j - 2u_{11} c \\ w_x = -2\lambda w + 2u_2 f + 2u_{10} h + 2u_{12} c. \end{cases} \tag{S2}$$

Supplement 3. The Leading Terms of (S2)

Take the initial values

$$a_0 = \alpha, \qquad b_0 = \beta, \qquad c_0 = \gamma, \qquad d_0 = e_0 = \cdots = v_0 = w_0 = 0.$$

From (S2), the first few sets of $a, \ldots, w$ in Section 2.1 can be computed as follows

$$
\begin{aligned}
a_1 &= b_1 = c_1 = 0, \qquad d_1 = \frac{1}{2}\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha) \\
e_1 &= \frac{1}{2}\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \\
f_1 &= \frac{1}{2}\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha) \\
g_1 &= \frac{1}{2}\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma) \\
h_1 &= \frac{1}{2}\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta) \\
j_1 &= \frac{1}{2}\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma) \\
k_1 &= \frac{1}{2}u_1(\alpha + \gamma), \qquad l_1 = \frac{1}{2}u_2(\alpha + \gamma) \\
m_1 &= u_3\beta, \qquad o_1 = u_4\beta, \qquad p_1 = u_5\alpha, \qquad q_1 = u_6\alpha \\
r_1 &= \frac{1}{2}u_7(\alpha + \beta), \qquad s_1 = \frac{1}{2}u_8(\alpha + \beta) \\
t_1 &= \frac{1}{2}u_9(\beta + \gamma), \qquad u_1 = \frac{1}{2}u_{10}(\beta + \gamma) \\
v_1 &= u_{11}\gamma, \qquad w_1 = u_{12}\gamma \\
k_2 &= \frac{1}{4}u_{1x}(\alpha + \gamma) + \frac{1}{4}u_9\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha) \\
&\quad + \frac{1}{4}u_{11}\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha) \\
&\quad + \frac{1}{4}u_5\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma) \\
&\quad + \frac{1}{4}u_7\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma) \\
l_2 &= -\frac{1}{4}u_{2x}(\alpha + \gamma) + \frac{1}{4}u_{10}\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \\
&\quad + \frac{1}{4}u_6\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha) \\
&\quad + \frac{1}{4}u_{12}\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma) \\
&\quad + \frac{1}{4}u_8\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta) \\
m_2 &= \frac{1}{2}u_{3x}\beta + \frac{1}{2}u_7\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \\
&\quad + \frac{1}{2}u_9\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta)
\end{aligned}
$$

$$
\begin{aligned}
o_2 = & -\frac{1}{2}u_{4x}\beta + \frac{1}{2}u_8\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha) \\
& + \frac{1}{2}u_{10}\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma) \\
p_2 = & \frac{1}{2}u_{5x}\alpha + \frac{1}{2}u_7\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha) \\
& + \frac{1}{2}u_1\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha) \\
q_2 = & -\frac{1}{2}u_{6x}\alpha + \frac{1}{2}u_8\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \\
& + \frac{1}{2}u_2\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma) \\
r_2 = & \frac{1}{4}u_{7x}(\alpha + \beta) + \frac{1}{4}u_3\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha) \\
& + \frac{1}{4}u_5\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \\
& + \frac{1}{4}u_9\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha) \\
& + \frac{1}{4}u_1\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta) \\
s_2 = & -\frac{1}{4}u_{8x}(\alpha + \beta) + \frac{1}{4}u_6\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha) \\
& + \frac{1}{4}u_4\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \\
& + \frac{1}{4}u_{10}\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma) \\
& + \frac{1}{4}u_2\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma) \\
t_2 = & \frac{1}{4}u_{9x}(\beta + \gamma) + \frac{1}{4}u_1\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \\
& + \frac{1}{4}u_7\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma) \\
& + \frac{1}{4}u_{11}\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta) \\
& + \frac{1}{4}u_3\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma) \\
u_2 = & -\frac{1}{4}u_{10x}(\beta + \gamma) + \frac{1}{4}u_2\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha) \\
& + \frac{1}{4}u_8\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}u_4\partial^{-1}(u_2u_7+u_3u_{10}+u_9u_{12})(\gamma-\beta)\\
&+\frac{1}{4}u_{12}\partial^{-1}(u_1u_8+u_4u_9+u_{10}u_{11})(\beta-\gamma)\\
v_2 =\ &\frac{1}{2}u_{11x}\gamma+\frac{1}{2}u_1\partial^{-1}(u_1u_6+u_2u_{11}+u_8u_9)(\alpha-\gamma)\\
&+\frac{1}{2}u_9\partial^{-1}(u_1u_8+u_4u_9+u_{10}u_{11})(\beta-\gamma)\\
w_2 =\ &-\frac{1}{2}u_{12x}\gamma+\frac{1}{2}u_2\partial^{-1}(u_1u_{12}+u_2u_5+u_7u_{10})(\gamma-\alpha)\\
&+\frac{1}{2}u_{10}\partial^{-1}(u_2u_7+u_3u_{10}+u_9u_{12})(\gamma-\beta)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Supplement 4. The First Integrable System Induced by the Hierarchy (4)

Below is the integrable system induced by the integrable hierarchy (4) for the case $n = 1$.

$$
\begin{aligned}
u_{1t} =\ &\frac{1}{2}u_{1x}(\alpha+\gamma)+\frac{1}{2}u_9\partial^{-1}(u_1u_{10}+u_4u_7+u_5u_8)(\beta-\alpha)\\
&+\frac{1}{2}u_{11}\partial^{-1}(u_1u_{12}+u_2u_5+u_7u_{10})(\gamma-\alpha)\\
&+\frac{1}{2}u_5\partial^{-1}(u_1u_6+u_2u_{11}+u_8u_9)(\alpha-\gamma)\\
&+\frac{1}{2}u_7\partial^{-1}(u_1u_8+u_4u_9+u_{10}u_{11})(\beta-\gamma)\\
u_{2t} =\ &\frac{1}{2}u_{2x}(\alpha+\gamma)-\frac{1}{2}u_{10}\partial^{-1}(u_2u_9+u_3u_8+u_6u_7)(\alpha-\beta)\\
&-\frac{1}{2}u_6\partial^{-1}(u_1u_{12}+u_2u_5+u_7u_{10})(\gamma-\alpha)\\
&-\frac{1}{2}u_{12}\partial^{-1}(u_1u_6+u_2u_{11}+u_8u_9)(\alpha-\gamma)\\
&-\frac{1}{2}u_8\partial^{-1}(u_2u_7+u_3u_{10}+u_9u_{12})(\gamma-\beta)\\
u_{3t} =\ &u_{3x}\beta+u_7\partial^{-1}(u_2u_9+u_3u_8+u_6u_7)(\alpha-\beta)\\
&+u_9\partial^{-1}(u_2u_7+u_3u_{10}+u_9u_{12})(\gamma-\beta)\\
u_{4t} =\ &u_{4x}\beta-u_8\partial^{-1}(u_1u_{10}+u_4u_7+u_5u_8)(\beta-\alpha)\\
&-u_{10}\partial^{-1}(u_1u_8+u_4u_9+u_{10}u_{11})(\beta-\gamma)\\
u_{5t} =\ &u_{5x}\alpha+u_7\partial^{-1}(u_1u_{10}+u_4u_7+u_5u_8)(\beta-\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&+ u_1\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha)\\
u_{6t} &= u_{6x}\alpha - u_8\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta)\\
&\quad - u_2\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma)\\
u_{7t} &= \frac{1}{2}u_{7x}(\alpha + \beta) + \frac{1}{2}u_3\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha)\\
&\quad + \frac{1}{2}u_5\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta)\\
&\quad + \frac{1}{2}u_9\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha)\\
&\quad + \frac{1}{2}u_1\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta)\\
u_{8t} &= \frac{1}{2}u_{8x}(\alpha + \beta) - \frac{1}{2}u_6\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha)\\
&\quad - \frac{1}{2}u_4\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta)\\
&\quad - \frac{1}{2}u_{10}\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma)\\
&\quad - \frac{1}{2}u_2\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma)\\
u_{9t} &= \frac{1}{2}u_{9x}(\beta + \gamma) + \frac{1}{2}u_1\partial^{-1}(u_2u_9 + u_3u_8 + u_6u_7)(\alpha - \beta) \qquad \text{(S3)}\\
&\quad + \frac{1}{2}u_7\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma)\\
&\quad + \frac{1}{2}u_{11}\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta)\\
&\quad + \frac{1}{2}u_3\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma)\\
u_{10t} &= \frac{1}{2}u_{10x}(\beta + \gamma) - \frac{1}{2}u_2\partial^{-1}(u_1u_{10} + u_4u_7 + u_5u_8)(\beta - \alpha)\\
&\quad - \frac{1}{2}u_[8]\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha)\\
&\quad - \frac{1}{2}u_4\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta)\\
&\quad - \frac{1}{2}u_{12}\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma)\\
u_{11t} &= u_{11x}\gamma + u_1\partial^{-1}(u_1u_6 + u_2u_{11} + u_8u_9)(\alpha - \gamma)\\
&\quad + u_9\partial^{-1}(u_1u_8 + u_4u_9 + u_{10}u_{11})(\beta - \gamma)\\
u_{12t} &= u_{12x}\gamma - u_2\partial^{-1}(u_1u_{12} + u_2u_5 + u_7u_{10})(\gamma - \alpha)\\
&\quad - u_{10}\partial^{-1}(u_2u_7 + u_3u_{10} + u_9u_{12})(\gamma - \beta).
\end{aligned}
$$

Supplement 5. The Recurrence Relation of $V_{0,x} = [U_1, V_0]$

In the isospectral problem studied in Section 2.2, by choosing the spectral matrices (5) and (3), the stationary zero-curvature equation $V_{0,x} = [U_1, V_0]$ yields the following recurrence relation

$$\begin{cases} a_x = u_1'g - u_2'f + u_5'q - u_6'p + u_7'e - u_8'd \\ b_x = u_3'o - u_4'm - u_7'e + u_8'd + u_9'u - u_{10}'t \\ c_x = -u_1'g + u_2'f + u_9'u - u_{10}'t + u_{11}'w - u_{12}'v \\ d_x = 2\lambda d + u_1'j - u_4'r + u_5's - u_7'a + u_7'b - u_{10}'k \\ e_x = -2\lambda e - u_2'h + u_3's - u_6'r + u_8'a - u_8'b + u_9'l \\ f_x = 2\lambda f - u_1'a + u_1'c + u_5'l + u_7'h - u_{10}'r - u_{12}'k \\ g_x = -2\lambda g + u_2'a - u_2'c - u_6'k - u_8'j + u_9's + u_{11}'l \\ h_x = -u_1'e + u_3'u + u_8'f + u_9'w - u_{10}'m - u_{12}'t \\ j_x = u_2'd - u_4't - u_7'g + u_9'o - u_{10}'v + u_{11}'u \\ k_x = u_1'v + u_2'p - u_5'g + u_7't - u_9'd - u_{11}'f \\ l_x = -u_1'q - u_2'w + u_6'f - u_8'u + u_{10}'e + u_{12}'g \\ m_x = -2\lambda m - 2u_3'b + 2u_8'r - 2u_9'h \\ o_x = 2\lambda o + 2u_4'b - 2u_7's + 2u_{10}'j \\ p_x = 2\lambda p + 2u_1'k - 2u_5'a + 2u_7'r \\ q_x = -2\lambda q - 2u_2'l + 2u_6'a - 2u_8's \\ r_x = u_1't - u_3'd - u_5'e + u_7'm + u_8'p - u_9'f \\ s_x = -u_2'u + u_4'e + u_6'd - u_7'q - u_8'o + u_{10}'g \\ t_x = -2\lambda t + u_2'r - u_3'j + u_8'k - u_9'b - u_9'c - u_{11}'h \\ u_x = 2\lambda u - u_1's + u_4'h - u_7'l + u_{10}'b + u_{10}'c + u_{12}'j \\ v_x = -2\lambda v + 2u_2'k - 2u_9'j - 2u_{11}'c \\ w_x = 2\lambda w - 2u_1'l + 2u_{10}'h + 2u_{12}'c. \end{cases} \tag{S4}$$

Supplement 6. The Leading Terms of (S4)

Take the initial values

$$a_0 = \alpha, \qquad b_0 = \beta, \qquad c_0 = \gamma, \qquad d_0 = e_0 = \cdots = v_0 = w_0 = 0.$$

From (S4), the first few sets of $a, \ldots, w$ in Section 2.2 can be computed as follows

$$a_1 = b_1 = c_1 = 0, \quad d_1 = \frac{1}{2}u_7'(\alpha - \beta), \quad e_1 = \frac{1}{2}u_8'(\alpha - \beta), \quad f_1 = \frac{1}{2}u_1'(\alpha - \gamma)$$

$$
\begin{aligned}
g_1 &= \frac{1}{2}u_2'(\alpha-\gamma), \qquad h_1 = \frac{1}{2}\partial^{-1}(u_1'u_8' + u_3'u_{10}' + u_9'u_{12}')(\beta-\gamma)\\
j_1 &= -\frac{1}{2}\partial^{-1}(u_2'u_7' + u_4'u_9' + u_{10}'u_{11}')(\beta-\gamma)\\
k_1 &= \frac{1}{2}\partial^{-1}(-u_1'u_{11}' + u_2'u_5' - u_7'u_9')(\alpha+\gamma)\\
l_1 &= \frac{1}{2}\partial^{-1}(-u_1'u_6' + u_2'u_{12}' + u_8'u_{10}')(\alpha+\gamma), \qquad m_1 = -u_3'\beta\\
o_1 &= -u_4'\beta, \qquad p_1 = u_5'\alpha, \qquad q_1 = u_6'\alpha\\
r_1 &= \frac{1}{2}\partial^{-1}(-u_1'u_9' - u_3'u_7' + u_5'u_8')(\alpha+\beta)\\
s_1 &= \frac{1}{2}\partial^{-1}(u_2'u_{10}' + u_4'u_8' - u_6'u_7')(\alpha+\beta), \qquad t_1 = -\frac{1}{2}u_9'(\beta+\gamma)\\
u_1 &= -\frac{1}{2}u_{10}'(\beta+\gamma), \qquad v_1 = -u_{11}'\gamma, \qquad w_1 = -u_{12}'\gamma\\
d_2 &= \frac{1}{4}u_{7x}'(\alpha-\beta) + \frac{1}{4}u_1'\partial^{-1}(u_2'u_7' + u_4'u_9' + u_{10}'u_{11}')(\beta-\gamma)\\
&\quad + \frac{1}{4}u_{10}'\partial^{-1}(-u_1'u_{11}' + u_2'u_5' - u_7'u_9')(\alpha+\gamma)\\
&\quad + \frac{1}{4}u_4'\partial^{-1}(-u_1'u_9' - u_3'u_7' + u_5'u_8')(\alpha+\beta)\\
&\quad - \frac{1}{4}u_5'\partial^{-1}(u_2'u_{10}' + u_4'u_8' - u_6'u_7')(\alpha+\beta)\\
e_2 &= -\frac{1}{4}u_{8x}'(\alpha-\beta) - \frac{1}{4}u_2'\partial^{-1}(u_1'u_8' + u_3'u_{10}' + u_9'u_{12}')(\beta-\gamma)\\
&\quad + \frac{1}{4}u_9'\partial^{-1}(-u_1'u_6' + u_2'u_{12}' + u_8'u_{10}')(\alpha+\gamma)\\
&\quad - \frac{1}{6}u_4'\partial^{-1}(-u_1'u_9' - u_3'u_7' + u_5'u_8')(\alpha+\beta)\\
&\quad + \frac{1}{4}u_3'\partial^{-1}(u_2'u_{10}' + u_4'u_8' - u_6'u_7')(\alpha+\beta)\\
f_2 &= \frac{1}{4}u_{1x}'(\alpha-\gamma) - \frac{1}{4}u_7'\partial^{-1}(u_1'u_8' + u_3'u_{10}' + u_9'u_{12}')(\beta-\gamma)\\
&\quad + \frac{1}{4}u_{12}'\partial^{-1}(-u_1'u_{11}' + u_2'u_5' - u_7'u_9')(\alpha+\gamma)\\
&\quad - \frac{1}{4}u_5'\partial^{-1}(-u_1'u_6' + u_2'u_{12}' + u_8'u_{10}')(\alpha+\gamma)\\
&\quad + \frac{1}{4}u_{10}'\partial^{-1}(-u_1'u_9' - u_3'u_7' + u_5'u_8')(\alpha+\beta)\\
g_2 &= -\frac{1}{4}u_{2x}'(\alpha-\gamma) + \frac{1}{4}u_8'\partial^{-1}(u_2'u_7' + u_4'u_9' + u_{10}'u_{11}')(\beta-\gamma)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}u_6'\partial^{-1}(-u_1'u_{11}'+u_2'u_5'-u_7'u_9')(\alpha+\gamma)\\
&+\frac{1}{4}u_{11}'\partial^{-1}(-u_1'u_6'+u_2'u_{12}'+u_8'u_{10}')(\alpha+\gamma)\\
&+\frac{1}{4}u_9'\partial^{-1}(u_2'u_{10}'+u_4'u_8'-u_6'u_7')(\alpha+\beta)\\
m_2 = {}&\frac{1}{2}u_{3x}'\beta+\frac{1}{2}u_8'\partial^{-1}(-u_1'u_9'-u_3'u_7'+u_5'u_8')(\alpha+\beta)\\
&-\frac{1}{2}u_9'\partial^{-1}(u_1'u_8'+u_3'u_{10}'+u_9'u_{12}')(\beta-\gamma)\\
o_2 = {}&-\frac{1}{2}u_{4x}'\beta+\frac{1}{2}u_{10}'\partial^{-1}(u_2'u_7'+u_4'u_9'+u_{10}'u_{11}')(\beta-\gamma)\\
&+\frac{1}{2}u_7'\partial^{-1}(u_2'u_{10}'+u_4'u_8'-u_6'u_7')(\alpha+\beta)\\
p_2 = {}&\frac{1}{2}u_{5x}'\alpha-\frac{1}{2}u_1'\partial^{-1}(-u_1'u_{11}'+u_2'u_5'-u_7'u_9')(\alpha+\gamma)\\
&-\frac{1}{2}u_7'\partial^{-1}(-u_1'u_9'-u_3'u_7'+u_5'u_8')(\alpha+\beta)\\
q_2 = {}&-\frac{1}{2}u_{6x}'\alpha-\frac{1}{2}u_2'\partial^{-1}(-u_1'u_6'+u_2'u_{12}'+u_8'u_{10}')(\alpha+\gamma)\\
&-\frac{1}{2}u_8'\partial^{-1}(u_2'u_{10}'+u_4'u_8'-u_6'u_7')(\alpha+\beta)\\
t_2 = {}&\frac{1}{4}u_{9x}'(\beta+\gamma)+\frac{1}{4}u_8'\partial^{-1}(-u_1'u_{11}'+u_2'u_5'-u_7'u_9')(\alpha+\gamma)\\
&-\frac{1}{4}u_{11}'\partial^{-1}(u_1'u_8'+u_3'u_{10}'+u_9'u_{12}')(\beta-\gamma)\\
&+\frac{1}{4}u_3'\partial^{-1}(u_2'u_7'+u_4'u_9'+u_{10}'u_{11}')(\beta-\gamma)\\
&+\frac{1}{4}u_2'\partial^{-1}(-u_1'u_9'-u_3'u_7'+u_5'u_8')(\alpha+\beta)\\
u_2 = {}&-\frac{1}{4}u_{10x}'(\beta+\gamma)-\frac{1}{4}u_4'\partial^{-1}(u_1'u_8'+u_3'u_{10}'+u_9'u_{12}')(\beta-\gamma)\\
&+\frac{1}{4}u_{12}'\partial^{-1}(u_2'u_7'+u_4'u_9'+u_{10}'u_{11}')(\beta-\gamma)\\
&+\frac{1}{4}u_7'\partial^{-1}(-u_1'u_6'+u_2'u_{12}'+u_8'u_{10}')(\alpha+\gamma)\\
&+\frac{1}{4}u_1'\partial^{-1}(u_2'u_{10}'+u_4'u_8'-u_6'u_7')(\alpha+\beta)\\
v_2 = {}&\frac{1}{2}u_{11x}'\gamma+\frac{1}{2}u_2'\partial^{-1}(-u_1'u_{11}'+u_2'u_5'-u_7'u_9')(\alpha+\gamma)
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{1}{2}u'_9\partial^{-1}(u'_2u'_7 + u'_4u'_9 + u'_{10}u'_{11})(\beta-\gamma) \\
w_2 = & -\frac{1}{2}u'_{12x}\gamma - \frac{1}{2}u'_{10}\partial^{-1}(u'_1u'_8 + u'_3u'_{10} + u'_9u'_{12})(\beta-\gamma) \\
& + \frac{1}{2}u'_1\partial^{-1}(-u'_1u'_6 + u'_2u'_{12} + u'_8u'_{10})(\alpha+\gamma) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Supplement 7. The Integrable System Induced by the Hierarchy (6)

Below is the integrable system induced by the integrable hierarchy (6) for the case $n = 1$

$$
\begin{aligned}
u'_{1t} = & \frac{1}{2}u'_{1x}(\alpha-\gamma) - \frac{1}{2}u'_7\partial^{-1}(u'_1u'_8 + u'_3u'_{10} + u'_9u'_{12})(\beta-\gamma) \\
& + \frac{1}{2}u'_{12}\partial^{-1}(-u'_1u'_{11} + u'_2u'_5 - u'_7u'_9)(\alpha+\gamma) \\
& - \frac{1}{2}u'_5\partial^{-1}(-u'_1u'_6 + u'_2u'_{12} + u'_8u'_{10})(\alpha+\gamma) \\
& + \frac{1}{2}u'_{10}\partial^{-1}(-u'_1u'_9 - u'_3u'_7 + u'_5u'_8)(\alpha+\beta) \\
u'_{2t} = & \frac{1}{2}u'_{2x}(\alpha-\gamma) - \frac{1}{2}u'_8\partial^{-1}(u'_2u'_7 + u'_4u'_9 + u'_{10}u'_{11})(\beta-\gamma) \\
& + \frac{1}{2}u'_6\partial^{-1}(-u'_1u'_{11} + u'_2u'_5 - u'_7u'_9)(\alpha+\gamma) \\
& - \frac{1}{2}u'_{11}\partial^{-1}(-u'_1u'_6 + u'_2u'_{12} + u'_8u'_{10})(\alpha+\gamma) \\
& - \frac{1}{2}u'_9\partial^{-1}(u'_2u'_{10} + u'_4u'_8 - u'_6u'_7)(\alpha+\beta) \\
u'_{3t} = & -u'_{3x}\beta + \frac{1}{2}u'_8\partial^{-1}(-u'_1u'_9 - u'_3u'_7 + u'_5u'_8)(\alpha+\beta) \\
& + u'_9\partial^{-1}(u'_1u'_8 + u'_3u'_{10} + u'_9u'_{12})(\beta-\gamma) \\
u'_{4t} = & -u'_{4x}\beta + \frac{1}{2}u'_{10}\partial^{-1}(u'_2u'_7 + u'_4u'_9 + u'_{10}u'_{11})(\beta-\gamma) \\
& + u'_7\partial^{-1}(u'_2u'_{10} + u'_4u'_8 - u'_6u'_7)(\alpha+\beta) \\
u'_{5t} = & u'_{5x}\alpha - u'_1\partial^{-1}(-u'_1u'_{11} + u'_2u'_5 - u'_7u'_9)(\alpha+\gamma) \\
& - u'_7\partial^{-1}(-u'_1u'_9 - u'_3u'_7 + u'_5u'_8)(\alpha+\beta) \\
u'_{6t} = & u'_{6x}\alpha - \frac{1}{2}u'_2\partial^{-1}(-u'_1u'_6 + u'_2u'_{12} + u'_8u'_{10})(\alpha+\gamma) \\
& + u'_8\partial^{-1}(u'_2u'_{10} + u'_4u'_8 - u'_6u'_7)(\alpha+\beta)
\end{aligned}
$$

$$
\begin{aligned}
u'_{7t} = {} & \frac{1}{2}u'_{7x}(\alpha-\beta) + \frac{1}{2}u'_1\partial^{-1}(u'_2u'_7 + u'_4u'_9 + u'_{10}u'_{11})(\beta-\gamma) \\
& + \frac{1}{2}u'_{10}\partial^{-1}(-u'_1u'_{11} + u'_2u'_5 - u'_7u'_9)(\alpha+\gamma) \\
& + \frac{1}{2}u'_4\partial^{-1}(-u'_1u'_9 - u'_3u'_7 + u'_5u'_8)(\alpha+\beta) \\
& - \frac{1}{2}u'_5\partial^{-1}(u'_2u'_{10} + u'_4u'_8 - u'_6u'_7)(\alpha+\beta) \\
u'_{8t} = {} & \frac{1}{2}u'_{8x}(\alpha-\beta) + \frac{1}{2}u'_2\partial^{-1}(u'_1u'_8 + u'_3u'_{10} + u'_9u'_{12})(\beta-\gamma) \\
& - \frac{1}{2}u'_9\partial^{-1}(-u'_1u'_6 + u'_2u'_{12} + u'_8u'_{10})(\alpha+\gamma) \\
& + \frac{1}{2}u'_4\partial^{-1}(-u'_1u'_9 - u'_3u'_7 + u'_5u'_8)(\alpha+\beta) \\
& - \frac{1}{2}u'_3\partial^{-1}(u'_2u'_{10} + u'_4u'_8 - u'_6u'_7)(\alpha+\beta) \\
u'_{9t} = {} & -\frac{1}{2}u'_{9x}(\beta+\gamma) - \frac{1}{2}u'_8\partial^{-1}(-u'_1u'_{11} + u'_2u'_5 - u'_7u'_9)(\alpha+\gamma) \\
& + \frac{1}{2}u'_{11}\partial^{-1}(u'_1u'_8 + u'_3u'_{10} + u'_9u'_{12})(\beta-\gamma) \\
& - \frac{1}{2}u'_3\partial^{-1}(u'_2u'_7 + u'_4u'_9 + u'_{10}u'_{11})(\beta-\gamma) \\
& - \frac{1}{2}u'_2\partial^{-1}(-u'_1u'_9 - u'_3u'_7 + u'_5u'_8)(\alpha+\beta) \\
u'_{10t} = {} & -\frac{1}{2}u'_{10x}(\beta+\gamma) - \frac{1}{2}u'_4\partial^{-1}(u'_1u'_8 + u'_3u'_{10} + u'_9u'_{12})(\beta-\gamma) \\
& + \frac{1}{2}u'_{12}\partial^{-1}(u'_2u'_7 + u'_4u'_9 + u'_{10}u'_{11})(\beta-\gamma) \\
& + \frac{1}{2}u'_7\partial^{-1}(-u'_1u'_6 + u'_2u'_{12} + u'_8u'_{10})(\alpha+\gamma) \\
& + \frac{1}{2}u'_1\partial^{-1}(u'_2u'_{10} + u'_4u'_8 - u'_6u'_7)(\alpha+\beta) \\
u'_{11t} = {} & -u'_{11x}\gamma - u'_2\partial^{-1}(-u'_1u'_{11} + u'_2u'_5 - u'_7u'_9)(\alpha+\gamma) \\
& - u'_9\partial^{-1}(u'_2u'_7 + u'_4u'_9 + u'_{10}u'_{11})(\beta-\gamma) \\
u'_{12t} = {} & -u'_{12x}\gamma - u'_{10}\partial^{-1}(u'_1u'_8 + u'_3u'_{10} + u'_9u'_{12})(\beta-\gamma) \\
& + u'_1\partial^{-1}(-u'_1u'_6 + u'_2u'_{12} + u'_8u'_{10})(\alpha+\gamma).
\end{aligned}
$$

Supplement 8. The recurrence relation of $V_{3,x} = [U_3, V_3]$

In the isospectral problem studied in Section 3, after constructing the spectral matrices U_3, V_3 via the Kronecker product, the stationary zero-curvature equation $V_{3,x} = [U_3, V_3]$ yields equation (S2) and the following recurrence relation

$$\begin{cases} a_x^* = u_1 l^* - u_2 k^* + u_5 q^* - u_6 p^* + u_7 s^* - u_8 r^* + u_1^* l - u_2^* k \\ \qquad +u_5^* q - u_6^* p + u_7^* s - u_8^* r \\ b_x^* = u_3 o^* - u_4 m^* + u_7 s^* - u_8 r^* + u_9 u^* - u_{10} t^* + u_3^* o - u_4^* m \\ \qquad +u_7^* s - u_8^* r + u_9^* u - u_{10}^* t \\ c_x^* = u_1 l^* - u_2 k^* + u_9 u^* - u_{10} t^* + u_{11} w^* - u_{12} v^* + u_1^* l - u_2^* k \\ \qquad +u_9^* u - u_{10}^* t + u_{11}^* w - u_{12}^* v \\ d_x^* = u_1 u^* - u_4 r^* + u_5 s^* + u_7 o^* - u_8 p^* - u_{10} k^* + u_1^* u - u_4^* r \\ \qquad +u_5^* s + u_7^* o - u_8^* p - u_{10}^* k \\ e_x^* = -u_2 t^* + u_3 s^* - u_6 r^* + u_7 q^* - u_8 m^* + u_9 l^* - u_2^* t + u_3^* s \\ \qquad -u_6^* r + u_7^* q - u_8^* m + u_9^* l \\ f_x^* = u_1 w^* - u_2 p^* + u_5 l^* + u_7 u^* - u_{10} r^* - u_{12} k^* + u_1^* w - u_2^* p \\ \qquad +u_5^* l + u_7^* u - u_{10}^* r - u_{12}^* k \\ g_x^* = u_1 q^* - u_2 v^* - u_6 k^* - u_8 t^* + u_9 s^* + u_{11} l^* + u_1^* q - u_2^* v \\ \qquad -u_6^* k - u_8^* t + u_9^* s + u_{11}^* l \\ h_x^* = -u_2 r^* + u_3 u^* + u_7 l^* + u_9 w^* - u_{10} m^* - u_{12} t^* - u_2^* r + u_3^* u \\ \qquad +u_7^* l + u_9^* w - u_{10}^* m - u_{12}^* t \\ j_x^* = u_1 s^* - u_4 t^* - u_8 k^* + u_9 o^* - u_{10} v^* + u_{11} u^* + u_1^* s - u_4^* t \\ \qquad -u_8^* k + u_9^* o - u_{10}^* v + u_{11}^* u \\ k_x^* = 2\lambda k^* - u_1 a^* - u_1 c^* - u_5 g^* - u_7 j^* - u_9 d^* - u_{11} f^* - u_1^* a \\ \qquad -u_1^* c - u_5^* g - u_7^* j - u_9^* d - u_{11}^* f \\ l_x^* = -2\lambda l^* + u_2 a^* + u_2 c^* + u_6 f^* + u_8 h^* + u_{10} e^* + u_{12} g^* + u_2^* a \\ \qquad +u_2^* c + u_6^* f + u_8^* h + u_{10}^* e + u_{12}^* g \\ m_x^* = 2\lambda m^* - 2u_3 b^* - 2u_7 e^* - 2u_9 h^* - 2u_3^* b - 2u_7^* e - 2u_9^* h \\ o_x^* = -2\lambda o^* + 2u_4 b^* + 2u_8 d^* + 2u_{10} j^* + 2u_4^* b + 2u_8^* d + 2u_{10}^* j \\ p_x^* = 2\lambda p^* - 2u_1 f^* - 2u_5 a^* - 2u_7 d^* - 2u_1^* f - 2u_5^* a - 2u_7^* d \\ q_x^* = -2\lambda q^* + 2u_2 g^* + 2u_6 a^* + 2u_8 e^* + 2u_2^* g + 2u_6^* a + 2u_8^* e \\ r_x^* = 2\lambda r^* - u_1 h^* - u_3 d^* - u_5 e^* - u_7 a^* - u_7 b^* - u_9 f^* - u_1^* h \\ \qquad -u_3^* d - u_5^* e - u_7^* a - u_7^* b - u_9^* f \end{cases} \tag{S5}$$

$$\begin{cases} s^*_x = -2\lambda s^* + u_2 j^* + u_4 e^* + u_6 d^* + u_8 a^* + u_8 b^* + u_{10} g^* + u^*_2 j \\ \qquad +u^*_4 e + u^*_6 d + u^*_8 a + u^*_8 b + u^*_{10} g \\ t^*_x = 2\lambda t^* - u_1 e^* - u_3 j^* - u_7 g^* - u_9 b^* - u_9 c^* - u_{11} h^* \\ \qquad -u^*_1 e - u^*_3 j + u^*_4 h - u^*_7 g - u^*_9 b - u^*_9 c - u^*_{11} h \\ u^*_x = -2\lambda u^* + u_2 d^* + u_4 h^* + u_8 f^* + u_{10} b^* + u_{10} c^* + u_{12} j^* \\ \qquad +u^*_2 d + u^*_8 f + u^*_{10} b + u^*_{10} c + u^*_{12} j \\ v^*_x = 2\lambda v^* - 2u_1 g^* - 2u_9 j^* - 2u_{11} c^* - 2u^*_1 g - 2u^*_9 j - 2u^*_{11} c \\ w^*_x = -2\lambda w^* + 2u_2 f^* + 2u_{10} h^* + 2u_{12} c^* + 2u^*_2 f + 2u^*_{10} h + 2u^*_{12} c. \end{cases}$$