



SYMMETRY, GEOMETRY AND QUANTIZATION WITH HYPERCOMPLEX NUMBERS

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Abstract. These notes describe some links between the group $SL_2(\mathbb{R})$, the Heisenberg group and hypercomplex numbers—complex, dual and double numbers. Relations between quantum and classical mechanics are clarified in this framework. In particular, classical mechanics can be obtained as a theory with *noncommutative* observables and a *non-zero* Planck constant if we replace complex numbers in quantum mechanics by dual numbers. Our consideration is based on induced representations which are build from complex-/dual-/double-valued characters. Dynamic equations, rules of additions of probabilities, ladder operators and uncertainty relations are also discussed. Finally, we prove a Calderón–Vaillancourt-type norm estimation for relative convolutions.

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